Summary of unbalanced pressure statistics

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/home/ross/Papers/DIAMET/Paper4/Report/PreliminaryResults.lyx

1 The problem

Variational data assimilation schemes rely on geostrophic and hydrostatic balances being valid. The key motivation behind this work is that at small scales the validity of these balances is questionable. In this report, all symbols have their usual meanings. For convenience we shall use the following abbreviations and convections:

- GB/gb - geostrophic balanced, geostrophically balanced, or linear balance,
- HB/hb - hydrostatic balanced or hydrostatically balanced,
- LS - linearisation state (subscript 0).
- LBE - linear balance equation.
- CS - convective-scale.
- LBC - lateral boundary condition.
- Perturbations are indicated with a δ.

2 The current Met Office scheme

2.1 The $\mathbf{T}_{\mathrm{p}}$-transform

The $\mathbf{T}_{\mathrm{p}}$-transform performs model perturbations to control parameter perturbations.

- The (input) model perturbations considered here: $\delta\vec{v}_{\mathrm{h}} = (u_{\mathrm{h}}, v_{\mathrm{h}})$ (horizontal wind), δp (pressure), and $\delta\theta$ (potential temperature). [Discussion is not made of humidity.]
- The (output) control parameter perturbations considered here: $\delta\psi$ (stream-function), and $\delta p^{\overline{\mathrm{lb}}}$ (linearly unbalanced pressure). [Discussion is not made of humidity nor velocity potential.]
- Find the HB pressure, δp^{hb}, by integrating (1) from the ground upwards:

$$\frac{R_{\mathrm{d}}}{c_p}\frac{\partial}{\partial z}\left[\left(\frac{p_0^{\mathrm{hb}}}{p_{1000}}\right)^{\frac{R_{\mathrm{d}}}{c_p}}\frac{\delta p^{\mathrm{hb}}}{p_0^{\mathrm{hb}}}\right] - \frac{g}{\theta_0^{\mathrm{hb}2}}\delta\theta^{\mathrm{hb}} = 0. \tag{1}$$

  - $\delta\theta^{\mathrm{hb}}$ is the hydrostatically balanced potential temperature increment, which is substituted by $\delta\theta$.
  - p_0^{hb} and θ_0^{hb} are the hydrostatically balanced LS pressure and potential temperature which are substituted by p_0 and θ_0 respectively.
  - We represent the solution in the form: $\delta p^{\mathrm{hb}} = (\mathbf{L}^{\mathrm{hb}})^{-1}\delta\theta^{\mathrm{hb}}$.

- Find the stream function $\delta\psi = \nabla_{\mathrm{h}}^{-2}\{\vec{k}\cdot\nabla\times\delta\vec{v}_{\mathrm{h}}\}$.

– $\vec{k}$ is the vertical unit vector.
– We represent the solution in the form: $\delta\psi = \mathbf{H}^{\psi v_h}\delta\vec{v}_{\mathrm{h}}$.

- Find GB pressure by solving (2) for δp^{lb} given $\delta\psi$:

$$\nabla_{\mathrm{h}}^2 \delta p^{\mathrm{lb}} + \nabla_{\mathrm{h}} \cdot \left(\frac{f}{\alpha_0^{\mathrm{lb}}} \nabla_{\mathrm{h}} \delta\psi \right) = 0. \tag{2}$$

  – The global mean of δp^{lb} is arbitrary. We set it to the same value as that of δp^{hb}. See footnote[1].
  – We represent the solution in the form: $\delta p^{\mathrm{lb}} = \mathbf{L}^{\mathrm{lb}}\delta\psi$.

- Perform a regression operation on δp^{lb} with a vertical regression operator, $\mathbf{R}$: $\delta p^{\mathrm{lbvr}} = \mathbf{R}\delta p^{\mathrm{lb}}$, where $\mathbf{R} = \mathbf{C}^{p^{\mathrm{hb}}p^{\mathrm{lb}}} \left(\mathbf{C}^{p^{\mathrm{lb}}p^{\mathrm{lb}}} \right)^{-1}$. See footnote[2]

- Find the residual pressure, $\delta p^{\overline{\mathrm{lb}}} = \delta p^{\mathrm{hb}} - \delta p^{\mathrm{lbvr}}$, using the above. If δp^{lbvr} is the true linearly balanced pressure then the residual is the true unbalanced pressure. Section 3 discusses how we may expect this to go wrong at convective scales.

The fields of control parameters ($\delta\psi$, $\delta p^{\overline{\mathrm{lb}}}$) (plus velocity potential, $\delta\chi$, and moisture, $\delta\mu$, in the full scheme) are used to calibrate their spatial background error covariance matrices, which have square-roots $\mathbf{B}_{\mathrm{s},\psi}^{1/2}$ and $\mathbf{B}_{\mathrm{s},p^{\overline{\mathrm{lb}}}}^{1/2}$ respectively.

2.2 The $\mathrm{U_p}$-transform

The $\mathbf{U}_{\mathrm{p}}$-transform is the inverse of the $\mathbf{T}_{\mathrm{p}}$-transform and is run for every Var iteration. The essential steps for calculating the model space parameters involves the following. The (assumed) completely uncorrelated control variables are $\delta\hat{\psi}$ and $\delta\hat{p}^{\overline{\mathrm{lb}}}$, which become control parameters via the spatial transforms, i.e. $\delta\psi = \mathbf{B}_{\mathrm{s},\psi}^{1/2}\delta\hat{\psi}$ and $\delta p^{\overline{\mathrm{lb}}} = \mathbf{B}_{\mathrm{s},p^{\overline{\mathrm{lb}}}}^{1/2}\delta\hat{p}^{\overline{\mathrm{lb}}}$.

- Compute the horizontal wind increments from $\delta\psi$ (and $\delta\chi$ in the full scheme), $\delta\vec{v}_{\mathrm{h}} = (\mathbf{H}^{\psi v_h})^{-1}\delta\psi$, where $(\mathbf{H}^{\psi v_h})^{-1}$is the Helmholtz operator that computes wind from streamfunction.
- Calculate the balanced pressure, $\delta p^{\mathrm{lb}} = \mathbf{L}^{\mathrm{lb}}\delta\psi$.
- Perform the vertical regression: $\delta p^{\mathrm{lbvr}} = \mathbf{R}\delta p^{\mathrm{lb}}$.
- Calculate the total pressure, $\delta p^{\mathrm{hb}} = \delta p^{\mathrm{lbvr}} + \delta p^{\overline{\mathrm{lb}}}$.
- Calculate the HB potential temperature found from $\delta\theta^{\mathrm{hb}} = \mathbf{L}^{\mathrm{hb}}\delta p^{\mathrm{hb}}$.

In matrix form part of the $\mathbf{U}$-transform is:

$$\begin{pmatrix} \delta\mathbf{v}_{\mathrm{h}} \\ \delta p^{\mathrm{hb}} \\ \delta\theta^{\mathrm{hb}} \end{pmatrix} = \begin{pmatrix} \mathbf{I} & 0 \\ 0 & \mathbf{I} \\ 0 & \mathbf{L}^{\mathrm{hb}} \end{pmatrix} \begin{pmatrix} (\mathbf{H}^{\psi v_h})^{-1} & 0 \\ \mathbf{R}\mathbf{L}^{\mathrm{lb}} & \mathbf{I} \end{pmatrix} \begin{pmatrix} \mathbf{B}_{\mathrm{s},\psi}^{1/2} & 0 \\ 0 & \mathbf{B}_{\mathrm{s},p^{\overline{\mathrm{lb}}}}^{1/2} \end{pmatrix} \begin{pmatrix} \delta\hat{\psi} \\ \delta\hat{p}^{\overline{\mathrm{lb}}} \end{pmatrix}, \tag{3}$$

$$\Uparrow \tag{4}$$

and the HB pressure and potential temperature variables are assumed to represent the total pressure and potential temperature perturbations (no hydrostatically unbalanced contributions). The vector at the point in the matrix equation indicated by the $\Uparrow$ is $\begin{pmatrix} \delta\vec{v}_{\mathrm{h}} & \delta p^{\mathrm{hb}} \end{pmatrix}$ and the step to the left is the calculation of HB potential temperature.

[1] Note that the linearly balanced pressure fields in the results part of this report are found in a different way to that used in Var. In Var. the perturbations are transformed to spectral space, where the Laplacian in (2) is easier to invert. This relies on the LB and HB pressure solutions satisfying Dirichlet LBCs. Our main training data set is an ensemble, where the members do not have the same LBCs. The pressure perturbations within the ensemble will not therefore satisfy Dirichelet LBCs. We solve the LBE with a GCR solver (no preconditioning) where the imposed LBCs of δp^{lb} are taken from δp^{hb} for each member.

[2] We have options for choice of the regression $\mathbf{R}$. One option is to use $\mathbf{I}$ (no vertical regression), another is to use the matrix used operationally, and yet another is to derive $\mathbf{R}$ from the data considered in this report. There is also choice concerning the inner product to use in this derivation. We derive $\mathbf{R}$ from the data and use $\mathbf{I}$ as the inner product.

By design, control variables each have a background variance of unity, $\left\langle \delta\hat{\psi}\delta\hat{\psi}^{\mathrm{T}} \right\rangle_{\mathrm{b}} = \mathbf{I}$ and $\left\langle \delta\hat{p}^{\overline{\mathrm{lb}}}\delta\hat{p}^{\overline{\mathrm{lb}}\mathrm{T}} \right\rangle_{\mathrm{b}} = \mathbf{I}$ (where $\left\langle \bullet \right\rangle_{\mathrm{b}}$means expectation over the background PDF), and are assumed to be uncorrelated, $\left\langle \delta\hat{\psi}\delta\hat{p}^{\overline{\mathrm{lb}}\mathrm{T}} \right\rangle_{\mathrm{b}} \sim 0$. For our purposes, we consider that the non-correlation property will be true only if $\mathbf{L}^{\mathrm{lb}}\delta\hat{\psi}$ is an exclusively balanced pressure (which is related to different physical processes than those related to $\delta\hat{p}^{\overline{\mathrm{lb}}}$). $\mathbf{L}^{\mathrm{lb}}$ is not be expected to be a perfect balance operator in CS systems.

3 What can go wrong with this at the convective scale?

The implied background error covariance matrix of the model perturbations is, using the above control variable statistics:

$$\begin{aligned}
\mathbf{B}^{\mathrm{imp}} &= \left\langle \begin{pmatrix} \delta\vec{v}_{\mathrm{h}} \\ \delta p \\ \delta\theta \end{pmatrix} \begin{pmatrix} \delta\vec{v}_{\mathrm{h}}^{\mathrm{T}} & \delta p^{\mathrm{T}} & \delta\theta^{\mathrm{T}} \end{pmatrix} \right\rangle_{\mathrm{b}}, \\
&= \begin{pmatrix}
(\mathbf{H}^{\psi v_h})^{-1}\mathbf{B}_{\mathrm{s},\psi}(\mathbf{H}^{\psi v_h})^{-\mathrm{T}} & (\mathbf{H}^{\psi v_h})^{-1}\mathbf{B}_{\mathrm{s},\psi}(\mathbf{L}^{\mathrm{lb}})^{\mathrm{T}}\mathbf{R}^{\mathrm{T}} & (\mathbf{H}^{\psi v_h})^{-1}\mathbf{B}_{\mathrm{s},\psi}(\mathbf{L}^{\mathrm{lb}})^{\mathrm{T}}\mathbf{R}^{\mathrm{T}}(\mathbf{L}^{\mathrm{hb}})^{\mathrm{T}} \\
\mathbf{R}\mathbf{L}^{\mathrm{lb}}\mathbf{B}_{\mathrm{s},\psi}(\mathbf{H}^{\psi v_h})^{-\mathrm{T}} & \begin{matrix}\mathbf{R}\mathbf{L}^{\mathrm{lb}}\mathbf{B}_{\mathrm{s},\psi}(\mathbf{L}^{\mathrm{lb}})^{\mathrm{T}}\mathbf{R}^{\mathrm{T}} \\ +\mathbf{B}_{\mathrm{s},p^{\overline{\mathrm{lb}}}}\end{matrix} & \begin{matrix}\mathbf{R}\mathbf{L}^{\mathrm{lb}}\mathbf{B}_{\mathrm{s},\psi}(\mathbf{L}^{\mathrm{lb}})^{\mathrm{T}}\mathbf{R}^{\mathrm{T}}(\mathbf{L}^{\mathrm{hb}})^{\mathrm{T}} \\ +\mathbf{B}_{\mathrm{s},p^{\overline{\mathrm{lb}}}}(\mathbf{L}^{\mathrm{hb}})^{\mathrm{T}}\end{matrix} \\
\mathbf{L}^{\mathrm{hb}}\mathbf{R}\mathbf{L}^{\mathrm{lb}}\mathbf{B}_{\mathrm{s},\psi}(\mathbf{H}^{\psi v_h})^{-\mathrm{T}} & \begin{matrix}\mathbf{L}^{\mathrm{hb}}\mathbf{R}\mathbf{L}^{\mathrm{lb}}\mathbf{B}_{\mathrm{s},\psi}(\mathbf{L}^{\mathrm{lb}})^{\mathrm{T}}\mathbf{R}^{\mathrm{T}} \\ +\mathbf{L}^{\mathrm{hb}}\mathbf{B}_{\mathrm{s},p^{\overline{\mathrm{lb}}}}\end{matrix} & \begin{matrix}\mathbf{L}^{\mathrm{hb}}\mathbf{R}\mathbf{L}^{\mathrm{lb}}\mathbf{B}_{\mathrm{s},\psi}(\mathbf{L}^{\mathrm{lb}})^{\mathrm{T}}\mathbf{R}^{\mathrm{T}}(\mathbf{L}^{\mathrm{hb}})^{\mathrm{T}} \\ +\mathbf{L}^{\mathrm{hb}}\mathbf{B}_{\mathrm{s},p^{\overline{\mathrm{lb}}}}(\mathbf{L}^{\mathrm{hb}})^{\mathrm{T}}\end{matrix}
\end{pmatrix}. \qquad (5)
\end{aligned}$$

The implied background error variances of $\delta\vec{v}_{\mathrm{h}}$, δp and $\delta\psi$ are found from the diagonal elements of (5).

3.1 Anomalies due to inappropriate geostrophic balance

Highlighting some key quantities:

- The implied covariance of δp (actually δp^{hb}) is $\mathbf{B}^{\mathrm{imp}}_{\mathrm{s},p} = \mathbf{R}\mathbf{L}^{\mathrm{lb}}\mathbf{B}_{\mathrm{s},\psi}(\mathbf{L}^{\mathrm{lb}})^{\mathrm{T}}\mathbf{R}^{\mathrm{T}} + \mathbf{B}_{\mathrm{s},p^{\overline{\mathrm{lb}}}}$.
- The unbalanced pressure is $\delta p - \mathbf{R}\mathbf{L}^{\mathrm{lb}}\mathbf{B}^{1/2}_{\mathrm{s},\psi}\delta\hat{\psi}$ whose covariance is $\mathbf{B}_{\mathrm{s},p^{\overline{\mathrm{lb}}}}$.

3.1.1 Formula 1 for the anomalous part of the implied covariance

Developing the formula for the total (actually HB according to the scheme) pressure variance:

$$\begin{aligned}
\left\langle \delta p \delta p^{\mathrm{T}} \right\rangle_{\mathrm{b}} &= \left\langle (\delta p^{\mathrm{lb}} + \delta p^{\overline{\mathrm{lb}}})(\delta p^{\mathrm{lb}} + \delta p^{\overline{\mathrm{lb}}})^{\mathrm{T}} \right\rangle_{\mathrm{b}}, \\
&= \left\langle \left(\mathbf{R}\mathbf{L}^{\mathrm{lb}}\mathbf{B}^{1/2}_{\mathrm{s},\psi}\delta\hat{\psi} + \mathbf{B}^{1/2}_{\mathrm{s},p^{\overline{\mathrm{lb}}}}\delta\hat{p}^{\overline{\mathrm{lb}}}\right)\left(\mathbf{R}\mathbf{L}^{\mathrm{lb}}\mathbf{B}^{1/2}_{\mathrm{s},\psi}\delta\hat{\psi} + \mathbf{B}^{1/2}_{\mathrm{s},p^{\overline{\mathrm{lb}}}}\delta\hat{p}^{\overline{\mathrm{lb}}}\right)^{\mathrm{T}} \right\rangle_{\mathrm{b}}, \\
&= \mathbf{R}\mathbf{L}^{\mathrm{lb}}\mathbf{B}_{\mathrm{s},\psi}(\mathbf{L}^{\mathrm{lb}})^{\mathrm{T}}\mathbf{R}^{\mathrm{T}} + \mathbf{B}_{\mathrm{s},p^{\overline{\mathrm{lb}}}} + \mathbf{R}\mathbf{L}^{\mathrm{lb}}\mathbf{B}^{1/2}_{\mathrm{s},\psi}\left\langle \delta\hat{\psi}(\delta\hat{p}^{\overline{\mathrm{lb}}})^{\mathrm{T}} \right\rangle_{\mathrm{b}}\mathbf{B}^{\mathrm{T}/2}_{\mathrm{s},p^{\overline{\mathrm{lb}}}} + \mathbf{B}^{1/2}_{\mathrm{s},p^{\overline{\mathrm{lb}}}}\left\langle \delta\hat{p}^{\overline{\mathrm{lb}}}\delta\hat{\psi}^{\mathrm{T}} \right\rangle_{\mathrm{b}}\mathbf{B}^{\mathrm{T}/2}_{\mathrm{s},\psi}(\mathbf{L}^{\mathrm{lb}})^{\mathrm{T}}\mathbf{R}^{\mathrm{T}}, \\
&= \mathbf{B}^{\mathrm{imp}}_{\mathrm{s},p} + \mathbf{R}\mathbf{L}^{\mathrm{lb}}\mathbf{B}^{1/2}_{\mathrm{s},\psi}\left\langle \delta\hat{\psi}(\delta\hat{p}^{\overline{\mathrm{lb}}})^{\mathrm{T}} \right\rangle_{\mathrm{b}}\mathbf{B}^{\mathrm{T}/2}_{\mathrm{s},p^{\overline{\mathrm{lb}}}} + \mathbf{B}^{1/2}_{\mathrm{s},p^{\overline{\mathrm{lb}}}}\left\langle \delta\hat{p}^{\overline{\mathrm{lb}}}\delta\hat{\psi}^{\mathrm{T}} \right\rangle_{\mathrm{b}}\mathbf{B}^{\mathrm{T}/2}_{\mathrm{s},\psi}(\mathbf{L}^{\mathrm{lb}})^{\mathrm{T}}\mathbf{R}^{\mathrm{T}}. \qquad (6)
\end{aligned}$$

The last two terms are related to the anomalous covariance.

3.1.2 Formula 2 for the anomalous part of the implied covariance

Returning to the bullet points at the start of Sect. 3. Substituting the latter into the former:

$$\begin{aligned}
\mathbf{B}^{\mathrm{imp}}_{\mathrm{s},p} &= \mathbf{R}\mathbf{L}^{\mathrm{lb}}\mathbf{B}_{\mathrm{s},\psi}(\mathbf{L}^{\mathrm{lb}})^{\mathrm{T}}\mathbf{R}^{\mathrm{T}} + \mathbf{B}_{\mathrm{s},p^{\overline{\mathrm{lb}}}}, \\
&= \mathbf{R}\mathbf{L}^{\mathrm{lb}}\mathbf{B}_{\mathrm{s},\psi}(\mathbf{L}^{\mathrm{lb}})^{\mathrm{T}}\mathbf{R}^{\mathrm{T}} + \left\langle \left(\delta p - \mathbf{R}\mathbf{L}^{\mathrm{lb}}\mathbf{B}^{1/2}_{\mathrm{s},\psi}\delta\hat{\psi}\right)\left(\delta p - \mathbf{R}\mathbf{L}^{\mathrm{lb}}\mathbf{B}^{1/2}_{\mathrm{s},\psi}\delta\hat{\psi}\right)^{\mathrm{T}} \right\rangle_{\mathrm{b}}, \\
&= \mathbf{B}_{\mathrm{s},p} + 2\mathbf{R}\mathbf{L}^{\mathrm{lb}}\mathbf{B}_{\mathrm{s},\psi}(\mathbf{L}^{\mathrm{lb}})^{\mathrm{T}}\mathbf{R}^{\mathrm{T}} - \mathbf{R}\mathbf{L}^{\mathrm{lb}}\mathbf{B}^{1/2}_{\mathrm{s},\psi}\left\langle \delta\hat{\psi}\delta p^{\mathrm{T}} \right\rangle_{\mathrm{b}} - \left\langle \delta p\delta\hat{\psi}^{\mathrm{T}} \right\rangle_{\mathrm{b}}\mathbf{B}^{\mathrm{T}/2}_{\mathrm{s},\psi}(\mathbf{L}^{\mathrm{lb}})^{\mathrm{T}}\mathbf{R}^{\mathrm{T}}, \qquad (7)
\end{aligned}$$

where $\mathbf{B}_{\mathrm{s},p}$ is the actual background error covariance of pressure. This shows that $\mathbf{B}^{\mathrm{imp}}_{\mathrm{s},p}$ and $\mathbf{B}_{\mathrm{s},p}$ differ as long as $2\mathbf{R}\mathbf{L}^{\mathrm{lb}}\mathbf{B}_{\mathrm{s},\psi}(\mathbf{L}^{\mathrm{lb}})^{\mathrm{T}}\mathbf{R}^{\mathrm{T}} - \mathbf{R}\mathbf{L}^{\mathrm{lb}}\mathbf{B}^{1/2}_{\mathrm{s},\psi}\left\langle \delta\hat{\psi}\delta p^{\mathrm{T}} \right\rangle_{\mathrm{b}} - \left\langle \delta p\delta\hat{\psi}^{\mathrm{T}} \right\rangle_{\mathrm{b}}\mathbf{B}^{\mathrm{T}/2}_{\mathrm{s},\psi}(\mathbf{L}^{\mathrm{lb}})^{\mathrm{T}}\mathbf{R}^{\mathrm{T}} \neq 0$.

3.1.3 A 'perfect balance' operator

Suppose that we have an operator, $\mathbf{Q}$, that calculates the true balanced pressure from $\delta\psi$ (e.g. by applying a filter to eliminate unbalanced motions, followed by the linear balance operator).

- Let $\delta\hat{p}^{\mathrm{lb}*}$ be the true unbalanced pressure control variable.
- Let $\delta\hat{p}^{\overline{\mathrm{lb}*}}$ be the true unbalanced pressure, satisfying $\delta p = \mathbf{R}^{*}\mathbf{Q}\mathbf{B}^{1/2}_{\mathrm{s},\psi}\delta\hat{\psi} + \mathbf{B}^{1/2}_{\mathrm{s},\hat{p}^{\overline{\mathrm{lb}*}}}\delta\hat{p}^{\overline{\mathrm{lb}*}}$ ($\mathbf{R}^{*}$ is the regression operator applied to the truly balanced system).
- Now $\delta\hat{\psi}$ and $\delta\hat{p}^{\overline{\mathrm{lb}*}}$ are truly uncorrelated, $\left\langle \delta\hat{\psi}\delta\hat{p}^{\overline{\mathrm{lb}*}} \right\rangle_{\mathrm{b}} = 0$.

These definitions give $\left\langle \delta\hat{\psi}\delta p^{\mathrm{T}} \right\rangle_{\mathrm{b}} = \mathbf{B}^{\mathrm{T}/2}_{\mathrm{s},\psi}\mathbf{Q}^{\mathrm{T}}\mathbf{R}^{*\mathrm{T}}$, which can be substituted into (7):

$$\begin{aligned}
\mathbf{B}^{\mathrm{imp}}_{\mathrm{s},p} &= \mathbf{B}_{\mathrm{s},p} + 2\mathbf{R}\mathbf{L}^{\mathrm{lb}}\mathbf{B}_{\mathrm{s},\psi}(\mathbf{L}^{\mathrm{lb}})^{\mathrm{T}}\mathbf{R}^{\mathrm{T}} - \mathbf{R}\mathbf{L}^{\mathrm{lb}}\mathbf{B}^{1/2}_{\mathrm{s},\psi}\left\langle \delta\hat{\psi}\delta p^{\mathrm{T}} \right\rangle_{\mathrm{b}} - \left\langle \delta p\delta\hat{\psi}^{\mathrm{T}} \right\rangle_{\mathrm{b}}\mathbf{B}^{\mathrm{T}/2}_{\mathrm{s},\psi}(\mathbf{L}^{\mathrm{lb}})^{\mathrm{T}}\mathbf{R}^{\mathrm{T}}, \\
&= \mathbf{B}_{\mathrm{s},p} + 2\mathbf{R}\mathbf{L}^{\mathrm{lb}}\mathbf{B}_{\mathrm{s},\psi}(\mathbf{L}^{\mathrm{lb}})^{\mathrm{T}}\mathbf{R}^{\mathrm{T}} - \mathbf{R}\mathbf{L}^{\mathrm{lb}}\mathbf{B}_{\mathrm{s},\psi}\mathbf{Q}^{\mathrm{T}}\mathbf{R}^{*\mathrm{T}} - \mathbf{R}^{*}\mathbf{Q}\mathbf{B}_{\mathrm{s},\psi}(\mathbf{L}^{\mathrm{lb}})^{\mathrm{T}}\mathbf{R}^{\mathrm{T}}, \\
&= \mathbf{B}_{\mathrm{s},p} + \mathbf{R}\mathbf{L}^{\mathrm{lb}}\mathbf{B}_{\mathrm{s},\psi}(\mathbf{R}\mathbf{L}^{\mathrm{lb}})^{\mathrm{T}} + \mathbf{R}\mathbf{L}^{\mathrm{lb}}\mathbf{B}_{\mathrm{s},\psi}(\mathbf{L}^{\mathrm{lb}})^{\mathrm{T}}\mathbf{R}^{\mathrm{T}} - \mathbf{R}\mathbf{L}^{\mathrm{lb}}\mathbf{B}_{\mathrm{s},\psi}(\mathbf{R}^{*}\mathbf{Q})^{\mathrm{T}} - \mathbf{R}^{*}\mathbf{Q}\mathbf{B}_{\mathrm{s},\psi}(\mathbf{L}^{\mathrm{lb}})^{\mathrm{T}}\mathbf{R}^{\mathrm{T}}, \\
&= \mathbf{B}_{\mathrm{s},p} + \mathbf{R}\mathbf{L}^{\mathrm{lb}}\mathbf{B}_{\mathrm{s},\psi}(\mathbf{R}\mathbf{L}^{\mathrm{lb}} - \mathbf{R}^{*}\mathbf{Q})^{\mathrm{T}} + (\mathbf{R}\mathbf{L}^{\mathrm{lb}} - \mathbf{R}^{*}\mathbf{Q})\mathbf{B}_{\mathrm{s},\psi}(\mathbf{L}^{\mathrm{lb}})^{\mathrm{T}}\mathbf{R}^{\mathrm{T}}.
\end{aligned} \tag{8}$$

This result says how the implied pressure covariances can be mis-specified in the DA system when the balance operator used, $\mathbf{L}^{\mathrm{lb}}$, differs from the 'correct' operator, $\mathbf{Q}$. Note that, if $\mathbf{R}\mathbf{L}^{\mathrm{lb}} = \mathbf{R}^{*}\mathbf{Q}$ then $\mathbf{B}^{\mathrm{imp}}_{\mathrm{s},p} = \mathbf{B}_{\mathrm{s},p}$.

3.1.4 Some basic diagnostics and note on the solution of the LB pressure

From dynamical arguments we might expect that the anomalous covariance is largest at small scales where the linear balance assumption is less valid. Let us consider only the diagonal elements of the covariances (either in real or spectral spaces).

3.1.5 Diagnostics from a convective-scale ensemble (no vertical regression)

The following Figs have been computed from one 24-member convective-scale ensemble of 1-hour forecasts from the NDP model (Southern UK) with no vertical regression. A cold front is passing over the UK dividing the domain into two - behind the front (the NW part) and ahead of the front (the SE part). All plots are for model level 35. Fig. 1 shows pressure variances for δp^{hb}, δp^{lb} and $\delta p^{\overline{\mathrm{lb}}}$. There is more total pressure variance to behind the front (NW of the domain) than ahead. The balanced pressure variances look reasonable (similar in magnitude and in large-scale pattern to the total pressure variances). The unbalanced (residual) pressure variances are smaller in magnitude and scale.

Fig. 2 shows the $\left\langle \delta p^{\mathrm{lb}}\delta p^{\overline{\mathrm{lb}}} \right\rangle$ correlation, and the anomaly found from (6) (in absolute and relative terms). Deviations from zero in all of these panels indicate that the covariance model used in Var. is anomalous. The relative anomalies tend to be highest just behind the front at the N end of the domain, although there are also strong anomalies ahead of the front. The right-hand panel in particular shows that the scheme overestimates the true variance by nearly 100% in the region near the northern edge. The parts of the domain behind the front generally have positive anomaly (overestimating total variance) and the parts ahead of the front generally have negative anomaly (underestimating total variance).

Now we examine the diagnostics in spectral space to see the scales where the anomalies in the variance occurs. This is done by first transforming the ensemble perturbations into spectral space and then computing the variances. Fig. 3 (left) shows the variance of δp^{hb}, δp^{lb} and $\delta p^{\overline{\mathrm{lb}}}$ as a function of horizontal scale. Perhaps surprisingly the balanced and unbalanced pressure variances have a similar order of magnitude over most scales. The exceptions are above 200 km and below 10 km (I am ignoring the data below 3 km as this is too close to the grid scale to be reliable). At the largest scales the unbalanced pressure variance is about 1-2 orders of magnitude smaller than the balanced pressure (which seems reasonable). At the small scales though the unbalanced pressure variance is about 1 order of magnitude smaller than the balanced pressure variance. Fig. 3 (middle) shows the relative anomaly as a function of length-scale and shows an envelope of anomalies of alternating sign that are high at around 3 km and

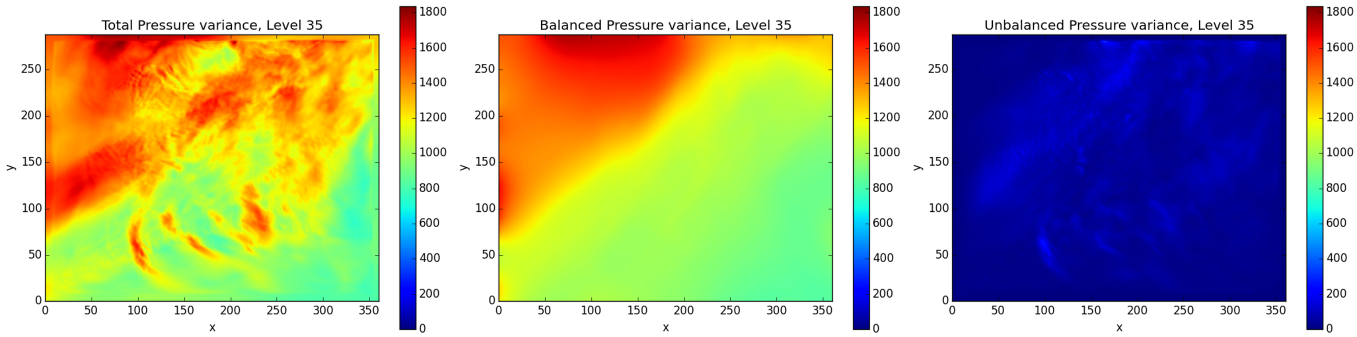

Figure 1: Ensemble-derived pressure variances for δp^{hb} (left panel), δp^{lb} (middle panel) and $\delta p^{\overline{\mathrm{lb}}}$ (right panel). Units: Pa^2. No vertical regression has been performed for these results.

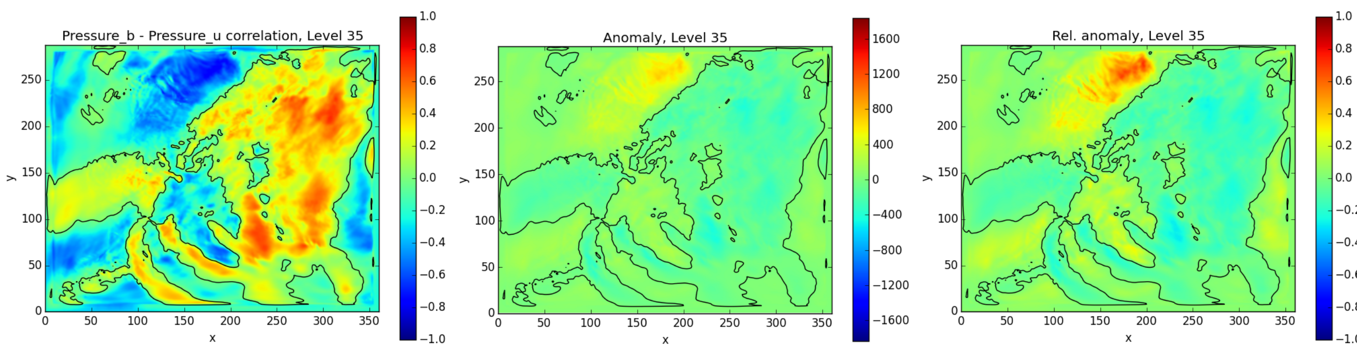

Figure 2: Ensemble-derived covariances between δp^{lb} and $\delta p^{\overline{\mathrm{lb}}}$ (left), the anomaly in the implied pressure variance (implied − total from (6) (middle), and the relative anomaly ((implied − total)/total, right). No vertical regression has been performed for these results.

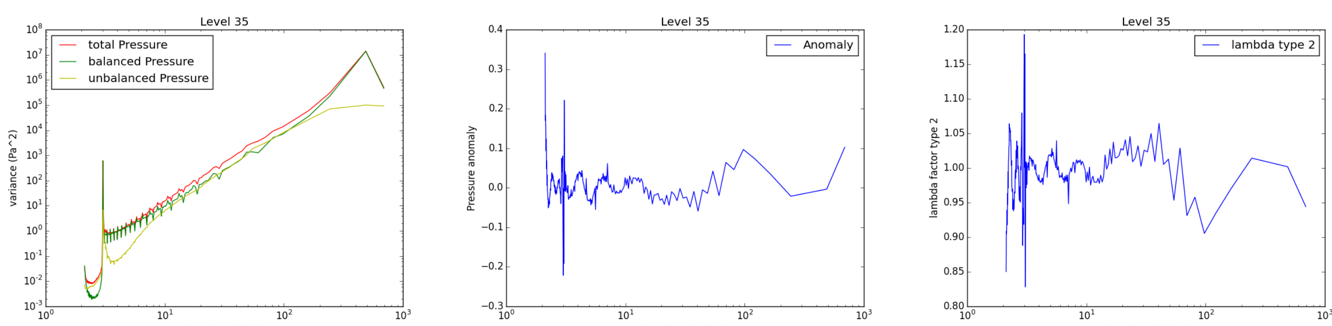

Figure 3: Ensemble-derived pressure diagnostics as a function of length-scale (derived from total wavenumber). Variances for δp^{hb}, δp^{lb} and $\delta p^{\overline{\mathrm{lb}}}$ (left panel), and relative anomaly (centre panel). No vertical regression has been performed for these results.

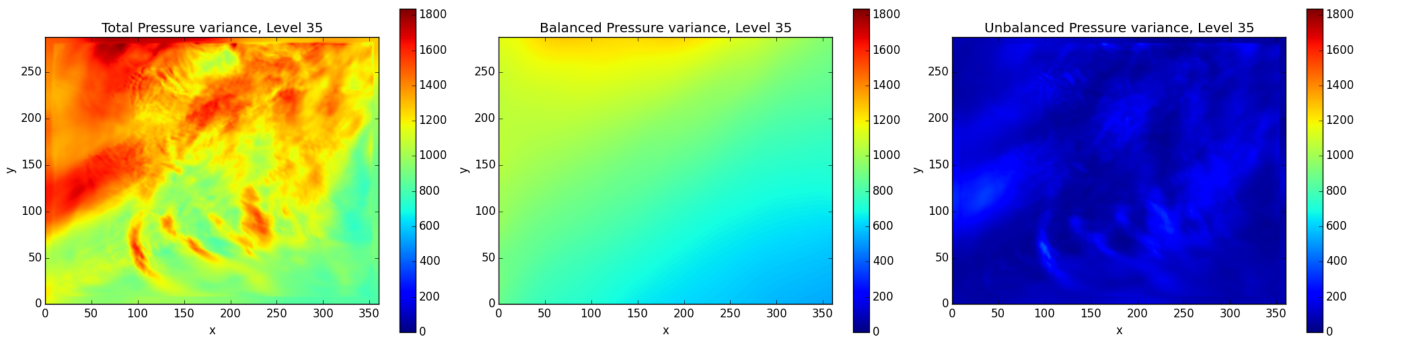

Figure 4: As Fig. 1 but for the case with vertical regression.

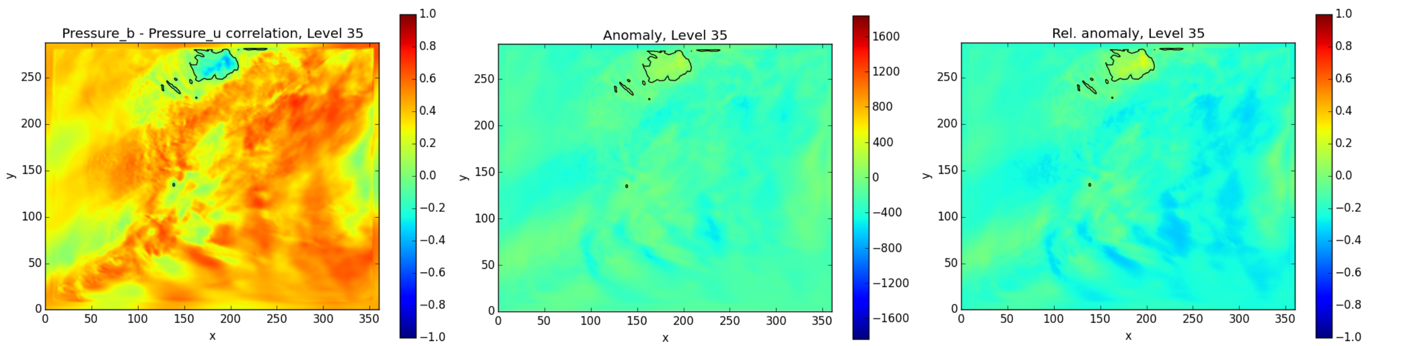

Figure 5: As Fig. 2 but for the case with vertical regression.

gradually decrease with increasing scale until about 40 km where they climb again. Note that a simplified version of (6) (diagonal elements only in either real or spectral space) is:

$$\begin{aligned} \sigma_p^2 &= \sigma_p^{\mathrm{imp}2} + 2\left\langle \delta p^{\mathrm{lb}} \delta p^{\overline{\mathrm{lb}}} \right\rangle_{\mathrm{b}}, \\ \text{anomaly} &= \text{implied} - \text{total}, \\ &= -2\left\langle \delta p^{\mathrm{lb}} \delta p^{\overline{\mathrm{lb}}} \right\rangle_{\mathrm{b}}. \end{aligned} \tag{9}$$

A positive anomaly therefore implies negative correlation between the balanced and unbalanced pressure perturbations.

3.1.6 Diagnostics from a convective-scale ensemble (with vertical regression)

Versions of the previous Figs., but for the case with vertical regression are shown in Figs. 4, 5 and 6. Note that we would expect this 'balanced pressure' (i.e. including the regression step) would give results that deviate from linear balance (even at LS). This is still a sensible thing to investigate though as vertical regression is done operationally.

Some discussion of these real-space results:

- The vertical regression reduces significantly the balanced pressure variances and increases the unbalanced pressure variances.
- The δp^{lb}-$\delta p^{\overline{\mathrm{lb}}}$ correlations have changed from regions of both sign correlation to mainly positive correlation.
- The anomaly has reduced.

Some discussion of these spectral-space results:

- The decrease (increase) in balanced (unbalanced) pressure variances is seen across all scales, and the unbalanced pressure variance is larger than the balanced pressure variance between about 8 and 200 km.

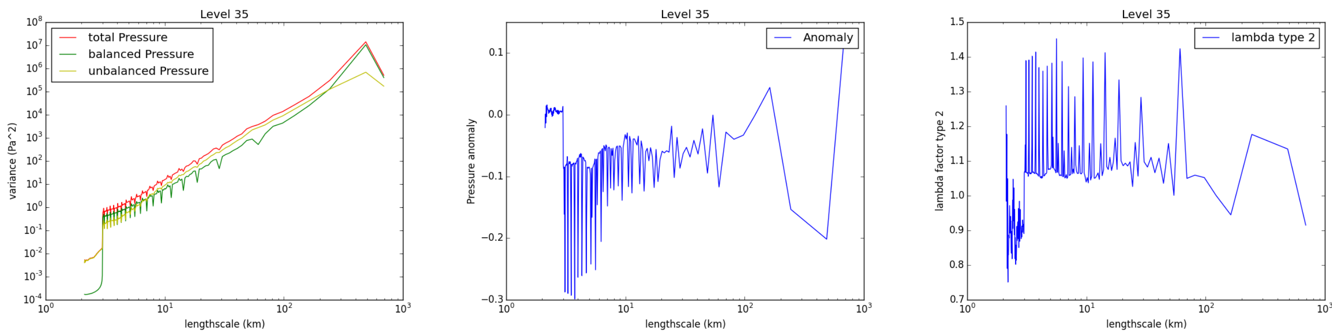

Figure 6: As Fig. 3but for the case with vertical regression.

- From 3 km upwards to 100 km, the anomaly follows an envelope that is negative and reduces towards zero (remember that, from (9)) a negative anomaly means positive correlation, which is consistent with the real-space results above).

3.1.7 A possible interpretation

We are investigating whether the LBE is appropriate or not. To try to understand the results above, the following data model is considered which accounts for imperfections in the definition of balance.

- Let the true degree of balance be a linear function of the diagnosed linear balance: $\delta p^{\mathrm{lb}*} = \tilde{\mathbf{\Lambda}} \delta p^{\mathrm{lb}}$, where recall that $\delta p^{\mathrm{lb}*}$ is the true balanced pressure perturbation. This implies that $\mathbf{R}^*\mathbf{Q} = \tilde{\mathbf{\Lambda}}\mathbf{RL}$. If we could estimate $\tilde{\mathbf{\Lambda}}$ from data then this may help improve data assimilation.

- Knowing $\delta p = \delta p^{\mathrm{lb}*} + \delta p^{\overline{\mathrm{lb}}*}$ and $\delta p = \delta p^{\mathrm{lb}} + \delta p^{\overline{\mathrm{lb}}}$, the model $\delta p^{\mathrm{lb}*} = \tilde{\mathbf{\Lambda}} \delta p^{\mathrm{lb}}$ (which is a relationship between the balanced perturbations) may be developed into a relationship between the unbalanced perturbations:

$$\begin{aligned} \delta p &= \tilde{\mathbf{\Lambda}} \delta p^{\mathrm{lb}} + \delta p^{\overline{\mathrm{lb}}*}, \\ &= \tilde{\mathbf{\Lambda}} (\delta p - \delta p^{\overline{\mathrm{lb}}}) + \delta p^{\overline{\mathrm{lb}}*}, \\ \Longrightarrow \delta p^{\overline{\mathrm{lb}}*} &= (1 - \tilde{\mathbf{\Lambda}}) \delta p + \tilde{\mathbf{\Lambda}} \delta p^{\overline{\mathrm{lb}}}. \end{aligned}$$

- The covariance between the true balanced and unbalanced pressure perturbations are then:

$$\begin{aligned} \left\langle \delta p^{\mathrm{lb}*} (\delta p^{\overline{\mathrm{lb}}*})^{\mathrm{T}} \right\rangle_{\mathrm{b}} &= \left\langle \left[\tilde{\mathbf{\Lambda}} \delta p^{\mathrm{lb}} \right] \left[(1 - \tilde{\mathbf{\Lambda}}) \delta p + \tilde{\mathbf{\Lambda}} \delta p^{\overline{\mathrm{lb}}} \right]^{\mathrm{T}} \right\rangle_{\mathrm{b}}, \\ &= \tilde{\mathbf{\Lambda}} \left\langle \delta p^{\mathrm{lb}} \delta p^{\mathrm{T}} \right\rangle_{\mathrm{b}} (1 - \tilde{\mathbf{\Lambda}})^{\mathrm{T}} + \tilde{\mathbf{\Lambda}} \left\langle \delta p^{\mathrm{lb}} (\delta p^{\overline{\mathrm{lb}}})^{\mathrm{T}} \right\rangle_{\mathrm{b}} \tilde{\mathbf{\Lambda}}^{\mathrm{T}}. \end{aligned}$$

- We define the true balance/unbalanced pressures as those that are mutually uncorrelated. The above then becomes:

$$0 = \tilde{\mathbf{\Lambda}} \left\langle \delta p^{\mathrm{lb}} \delta p^{\mathrm{T}} \right\rangle_{\mathrm{b}} (1 - \tilde{\mathbf{\Lambda}})^{\mathrm{T}} + \tilde{\mathbf{\Lambda}} \left\langle \delta p^{\mathrm{lb}} (\delta p^{\overline{\mathrm{lb}}})^{\mathrm{T}} \right\rangle_{\mathrm{b}} \tilde{\mathbf{\Lambda}}^{\mathrm{T}}.$$

- Suppose that in spectral space, the operator $\tilde{\mathbf{\Lambda}}$ (called $\mathbf{\Lambda}$ in spectral space) is diagonal. Let the diagonal element of $\mathbf{\Lambda}$ at a particular scale be λ. The variances of the above become:

$$0 = \lambda (1 - \lambda) \left\langle \delta p^{\mathrm{lb}} \delta p \right\rangle_{\mathrm{b}} + \lambda^2 \left\langle \delta p^{\mathrm{lb}} \delta p^{\overline{\mathrm{lb}}} \right\rangle_{\mathrm{b}}.$$

- One solution is $\lambda = 0$ (which says that non-correlation can be achieved by having zero balanced pressure; we ignore this solution). The other solution is:

$$\lambda = \frac{\left\langle \delta p^{\mathrm{lb}} \delta p \right\rangle_{\mathrm{b}}}{\left\langle \delta p^{\mathrm{lb}} \delta p \right\rangle_{\mathrm{b}} - \left\langle \delta p^{\mathrm{lb}} \delta p^{\overline{\mathrm{lb}}} \right\rangle_{\mathrm{b}}}. \tag{10}$$

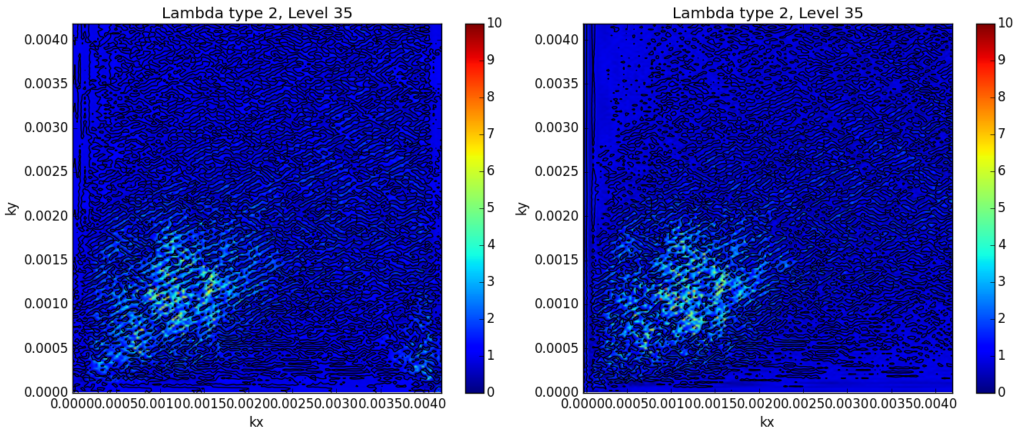

Figure 7: Diagnosed values of λ as a function of wavenumber for the 'no vertical regression' (left) and 'vertical regression' (right) cases.

Noting that $\left\langle \delta p^{\mathrm{lb}} \delta p^{\overline{\mathrm{lb}}} \right\rangle_{\mathrm{b}} = \left\langle \delta p^{\mathrm{lb}} (\delta p - \delta p^{\mathrm{lb}}) \right\rangle_{\mathrm{b}} = \left\langle \delta p^{\mathrm{lb}} \delta p \right\rangle_{\mathrm{b}} - \left\langle (\delta p^{\mathrm{lb}})^2 \right\rangle_{\mathrm{b}}$, the above becomes:

$$\lambda = \frac{\left\langle \delta p^{\mathrm{lb}} \delta p \right\rangle_{\mathrm{b}}}{\left\langle (\delta p^{\mathrm{lb}})^2 \right\rangle_{\mathrm{b}}}. \tag{11}$$

Here are my expectations.

- At large scales, balance is valid and we would expect $\lambda \approx 1$. This is consistent with (10) with $\left\langle \delta p^{\mathrm{lb}} \delta p^{\overline{\mathrm{lb}}} \right\rangle_{\mathrm{b}} = 0$.
- At small scales where the use of the LBE is less valid, the 'balanced pressure' calculated from the LBE will be incorrect. We might expect the balanced pressure to be systematically too large in magnitude and so the correction would satisfy $0 < \lambda < 1$ (λ would get progressively closer to zero with smaller scales).
- We would not expect $\lambda > 1$ or $\lambda < 0$.

Here is what we find from the data.

- Figure 3 (right panel) shows the diagnosed λ as a function of total wavenumber for the case with no vertical regression. The expected behaviour is not found! As before, ignore anything below 3km in scale. The value of λ oscillates around 1 with increasing amplitude with increasing scale. This breaks our expectations: (i) the envelope of deviations gets worse for larger scales, and values of λ often exceed 1.
- The value of λ does not go below zero though, as expected.
- Figure 6 (right panel) shows the diagnosed λ as a function of total wavenumber for the case with vertical regression. The values of λ are larger than for the no regression case and are almost always greater than 1. This is consistent with the above: regression will lessen balance and so λ will need to be larger to compensate.
- In order to resolve the total wavenumber results into each wave-vector, Fig. 7 plots λ as a function of $\vec{k} = (k_x, k_y)$ (the plots that are a function of total wavenumber, $k = \sqrt{k_x^2 + k_y^2}$, are an average of these data over all $\vec{k}$ that have similar values of k). In the cases without and with vertical regression, there is a region of large λ at values of k_x and k_y between about $0.0004\mathrm{m}^{-1}$ and $0.0020\mathrm{m}^{-1}$. This represents scales from about 15km down to 3km. Elsewhere λ is close to unity.
- This is virtually the opposite of what was expected. $\lambda > 1$ means that the true balance is actually greater than that calculated by the LBE at small scales!