ADI preconditioning for the PV equations

Ross Bannister, May 2006. Tidied December 2008.

1. The ADI Method

The equation,

$$\mathbf{A}\boldsymbol{x} = \boldsymbol{b}, \tag{1}$$

can be solved via a relaxation method by solving [1],

$$\lim_{t \to \infty} \frac{\partial \boldsymbol{x}}{\partial t} = \mathbf{A}\boldsymbol{x} - \boldsymbol{b}. \tag{2}$$

The operator $\mathbf{A}$ is assumed to have components associated with each direction in space (e.g. terms containing derivatives in x, y and z), allowing it to be written,

$$\mathbf{A} = \mathbf{A}_x + \mathbf{A}_y + \mathbf{A}_z. \tag{3}$$

The Alternating Direction Implicit (ADI) method works by dealing with Eq. (2) in an iterative fashion by considering each 'direction component' of $\mathbf{A}$ in turn, and treating each in an implicit way [2]. This is illustrated below for $\mathbf{A}$ in two and three dimensions.

2. Two-Dimensional ADI Method

Considering two dimensions to $\mathbf{A}$ (x and y), Eq. (2) can be discretised in time as follows,

$$\frac{\boldsymbol{x}^{i+1} - \boldsymbol{x}^{i}}{\delta t} = (\mathbf{A}_x + \mathbf{A}_y)\boldsymbol{x}^{i} - \boldsymbol{b}, \tag{4}$$

where the superscript denotes the time index. Equation (4) considers a time step of δt. In the following two steps - which are based on (4) - this time step is divided into two parts - one from $t \to t + \delta t_1$ and the other from $t + \delta t_1 \to t + \delta t_1 + \delta t_2$, where $\delta t = \delta t_1 + \delta t_2$. The components of $\mathbf{A}$ are treated differently in each part.

Step (i): treat $\mathbf{A}_x$ implicitly and $\mathbf{A}_y$ explicitly in (4), and advance by δt_1,

$$\frac{\boldsymbol{x}^{i+½} - \boldsymbol{x}^{i}}{\delta t_1} = \mathbf{A}_x\boldsymbol{x}^{i+½} + \mathbf{A}_y\boldsymbol{x}^{i} - \boldsymbol{b},$$

$$\left(\frac{1}{\delta t_1}\mathbf{I} - \mathbf{A}_x\right)\boldsymbol{x}^{i+½} = \left(\mathbf{A}_y + \frac{1}{\delta t_1}\mathbf{I}\right)\boldsymbol{x}^{i} - \boldsymbol{b}. \tag{5a}$$

Step (ii): treat $\mathbf{A}_y$ implicitly and $\mathbf{A}_x$ explicitly in (4), and advance a by δt_2,

$$\frac{\boldsymbol{x}^{i+1} - \boldsymbol{x}^{i+½}}{\delta t_2} = \mathbf{A}_x\boldsymbol{x}^{i+½} + \mathbf{A}_y\boldsymbol{x}^{i+1} - \boldsymbol{b},$$

$$\left(\frac{1}{\delta t_2}\mathbf{I} - \mathbf{A}_y\right)\boldsymbol{x}^{i+1} = \left(\mathbf{A}_x + \frac{1}{\delta t_2}\mathbf{I}\right)\boldsymbol{x}^{i+½} - \boldsymbol{b}. \tag{5b}$$

To advance each partial time step requires the inversion only of a one dimensional operator. This method can, in principle, be used to solve Eq. (1) by advancing this scheme for a large and sufficient number of time steps, but for preconditioning, only a few full-time-steps may be required for an approximate answer.

3. Three-Dimensional ADI Method

Equation (2) for three dimensions (x, y and z) becomes,

$$\frac{\boldsymbol{x}^{i+1} - \boldsymbol{x}^{i}}{\delta t} = (\mathbf{A}_x + \mathbf{A}_y + \mathbf{A}_z)\boldsymbol{x}^{i} - \boldsymbol{b}. \tag{6}$$

This is treated in three steps. Instead of treating each component in turn fully implicitly (with the remaining components explicitly) as is done in two dimensions, we treat each component in turn partially implicitly. Let there be three partial time steps, δt_a, δt_2 and δt_3, $\delta t_1 + \delta t_2 + \delta t_3 = \delta t$.

Step (i): treat $\mathbf{A}_x$ partially implicitly and $\mathbf{A}_y$ and $\mathbf{A}_z$ explicitly in (6), and advance over δt_1,

$$\frac{\boldsymbol{x}^{i+\frac{1}{3}} - \boldsymbol{x}^{i}}{\delta t_1} = \mathbf{A}_x(\varepsilon \boldsymbol{x}^{i+\frac{1}{3}} + (1-\varepsilon)\boldsymbol{x}^{i}) + (\mathbf{A}_y + \mathbf{A}_z)\boldsymbol{x}^{i} - \boldsymbol{b},$$

$$\left(\frac{1}{\delta t_1}\mathbf{I} - \varepsilon\mathbf{A}_x\right)\boldsymbol{x}^{i+\frac{1}{3}} = \left(\frac{1}{\delta t_1}\mathbf{I} + (1-\varepsilon)\mathbf{A}_x + \mathbf{A}_y + \mathbf{A}_z\right)\boldsymbol{x}^{i} - \boldsymbol{b}. \tag{7a}$$

ε is a parameter to determine the degree of implicitness in the treatment of the first direction component of $\mathbf{A}$ (where $\varepsilon = 1$ is fully implicit).

Step (ii): treat $\mathbf{A}_y$ partially implicitly and $\mathbf{A}_x$ and $\mathbf{A}_z$ explicitly in (6), and advance over δt_2,

$$\frac{\boldsymbol{x}^{i+\frac{2}{3}} - \boldsymbol{x}^{i+\frac{1}{3}}}{\delta t_2} = \mathbf{A}_y(\varepsilon \boldsymbol{x}^{i+\frac{2}{3}} + (1-\varepsilon)\boldsymbol{x}^{i+\frac{1}{3}}) + (\mathbf{A}_x + \mathbf{A}_z)\boldsymbol{x}^{i+\frac{1}{3}} - \boldsymbol{b},$$

$$\left(\frac{1}{\delta t_2}\mathbf{I} - \varepsilon\mathbf{A}_y\right)\boldsymbol{x}^{i+\frac{2}{3}} = \left(\frac{1}{\delta t_2}\mathbf{I} + (1-\varepsilon)\mathbf{A}_y + \mathbf{A}_x + \mathbf{A}_z\right)\boldsymbol{x}^{i+\frac{1}{3}} - \boldsymbol{b}. \tag{7b}$$

Step (iii): treat $\mathbf{A}_z$ partially implicitly and $\mathbf{A}_x$ and $\mathbf{A}_y$ explicitly in (6), and advance over δt_3,

$$\frac{\boldsymbol{x}^{i+1} - \boldsymbol{x}^{i+\frac{2}{3}}}{\delta t_3} = \mathbf{A}_z(\varepsilon \boldsymbol{x}^{i+1} + (1-\varepsilon)\boldsymbol{x}^{i+\frac{2}{3}}) + (\mathbf{A}_x + \mathbf{A}_y)\boldsymbol{x}^{i+\frac{2}{3}} - \boldsymbol{b},$$

$$\left(\frac{1}{\delta t_3}\mathbf{I} - \varepsilon\mathbf{A}_z\right)\boldsymbol{x}^{i+1} = \left(\frac{1}{\delta t_3}\mathbf{I} + (1-\varepsilon)\mathbf{A}_z + \mathbf{A}_x + \mathbf{A}_y\right)\boldsymbol{x}^{i+\frac{2}{3}} - \boldsymbol{b}. \tag{7c}$$

4. Application of ADI to balanced and unbalanced equations in PV control variable project

In the 'potential vorticity' control variable project, we have to solve two three-dimensional elliptic partial differential equations - one to compute the balanced component of the flow (represented by the balanced streamfunction parameter), and another to compute the unbalanced component (represented by the unbalanced pressure parameter). Each has a similar equation to solve. We solve these equations by the Generalised Conjugate Residual method (GCR), but wish to test-out the ADI method as a preconditioner within the GCR framework.

Even though the equations are three-dimensional, we shall use the two-dimensional ADI method by combining two operators (x and y) together, as shown in the following sections.

Equations (5a) and (5b) under these circumstances are,

$$\left(\kappa_A \mathbf{I} + \mathbf{A}_{xy}\right)\boldsymbol{x}^{i+½} = \boldsymbol{b} + \left(\kappa_A \mathbf{I} - \mathbf{A}_z\right)\boldsymbol{x}^{i}, \tag{8a}$$

$$\left(\kappa_B \mathbf{I} + \mathbf{A}_z\right)\boldsymbol{x}^{i+1} = \boldsymbol{b} + \left(\kappa_B \mathbf{I} - \mathbf{A}_{xy}\right)\boldsymbol{x}^{i+½}, \tag{8b}$$

where κ_A and κ_B are constants, and are akin to $-1/\delta t_1$ in Eq. (5a) and $-1/\delta t_2$ in Eq. (5b) respectively. This gets the ADI equations in the same form as used in [2]. In [2], κ_A and κ_B are the same (= κ) and κ is called an acceleration parameter. In [2] no mention is made of the sign of κ, although from the above argument, κ must be negative.

4.1 The balanced system

The ADI method will be used to approximate the solution of the following equation for the balanced streamfunction, ψ_B',

$$A_{xy}\psi_B' + \mathbf{A}_z\psi_B' = \frac{\rho_0}{\theta_{0z}}PV, \tag{9}$$

where,

$$\mathbf{A}_{xy}\psi_B' = \nabla_z^2\psi_B',$$

$$\mathbf{A}_z\psi_B' = \frac{\rho_0}{\theta_{0z}}\left(-\frac{f^2\theta_{0z}(1-\kappa)}{\rho_0 R\Pi_0\hat{\theta}_0}\psi_B' - \frac{f^2\theta_{0z}}{\rho_0\hat{\theta}_0}\overline{\frac{\theta_0}{\Pi_{0z}}\frac{\partial}{\partial z}\left(\kappa\frac{\Pi_0\rho_0}{p_0}\psi_B'\right)} + \right.$$
$$\left.\frac{f^2 g}{\rho_0 c_p\hat{\Pi}_{0z}^2}\frac{\partial^2}{\partial z^2}\left(\frac{\kappa\Pi_0\rho_0}{p_0}\psi_B'\right) - \frac{2f^2 g\Pi_{0zz}}{\rho_0 c_p\hat{\Pi}_{0z}^3}\overline{\frac{\partial}{\partial z}\left(\frac{\kappa\Pi_0\rho_0}{p_0}\psi_B'\right)}\right), \tag{10}$$

$\rho_0/\theta_{0z}\,PV$ is prescribed and is akin to $\boldsymbol{b}$. This equation is Eq. (3) in [3]. Substituting this information into Eq. (8a) gives,

$$\boxed{\left(\kappa_A\mathbf{I} + \nabla_z^2\right)\boldsymbol{x}^{i+½} = \frac{\rho_0}{\theta_{0z}}PV + \left(\kappa_A\mathbf{I} - \mathbf{A}_z\right)\boldsymbol{x}^{i}. \qquad (11a)}$$

This has the form of a screened Poisson equation [4] if $\kappa_A\rho_0/\theta_{0z}$ is negative (ρ_0/θ_{0z} is normally positive and κ_A is negative using the definition given after Eq. (8b)). This equation will require some attention to solve - see section 6 of this report.

Substituting Eqs. (10) into Eq. (8b) gives,

$$\boxed{\left(\kappa_B\mathbf{I} + \mathbf{A}_z\right)\boldsymbol{x}^{i+1} = \frac{\rho_0}{\theta_{0z}}PV + \left(\kappa_B\mathbf{I} - \nabla_z^2\right)\boldsymbol{x}^{i+½}. \qquad (11b)}$$

This equation is easy to solve as it involves only a tridiagonal equation to solve in the vertical.

4.2 The unbalanced system

The ADI method will be used to approximate the solution of the following equation for the unbalanced pressure, ψ_B',

$$\nabla^2 p_u' + \mathbf{A}_z p_u' = -\overline{PV}, \tag{12}$$

where

$$\mathbf{A}_{xy}p'_u = \nabla_z^2 p'_u,$$

$$\mathbf{A}_z p'_u = -\frac{f^2(1-\kappa)}{R\Pi_0\hat{\theta}_0}p'_u - \frac{f^2\rho_0}{\hat{\theta}_0}\overline{\frac{\theta_0}{\Pi_{0z}}\frac{\partial}{\partial z}\left(\kappa\frac{\Pi_0}{p_0}p'_u\right)} +$$

$$\frac{f^2 g\rho_0}{c_p\theta_{0z}\hat{\Pi}_{0z}^2}\frac{\partial^2}{\partial z^2}\left(\frac{\kappa\Pi_0}{p_0}p'_u\right) - \frac{2f^2 g\Pi_{0zz}\rho_0}{c_p\theta_{0z}\hat{\Pi}_{0z}^3}\overline{\frac{\partial}{\partial z}\left(\frac{\kappa\Pi_0}{p_0}p'_u\right)}. \qquad (13)$$

$-\overline{PV}$ is prescribed and is akin to ***b***. This equation is Eq. (22) in [3]. Substituting Eqs. (13) into Eq. (8a) gives,

$$\boxed{\left(\kappa_B\mathbf{I} + \nabla_z^2\right)\boldsymbol{x}^{i+½} = -\overline{PV} + \left(\kappa_B\mathbf{I} - \mathbf{A}_z\right)\boldsymbol{x}^i. \qquad (14a)}$$

As with the balanced system, this is a screened Poisson equation by the definition in [4] - see section 6 of this report.

Substituting Eqs. (13) into Eq. (8b) gives,

$$\boxed{\left(\kappa_B\mathbf{I} + \mathbf{A}_z\right)\boldsymbol{x}^{i+1} = -\overline{PV} + \left(\kappa_B\mathbf{I} - \nabla_z^2\right)\boldsymbol{x}^{i+½}. \qquad (14b)}$$

5. Discretised forms of Az

Ref [3] gives the discretised forms of $\mathbf{A}_z$ for the balanced and unbalanced systems. These are given here for completeness. In the following equations, all quantities are at level k unless otherwise indicated.

5.1 The balanced system

$$\mathbf{A}_z\psi'_B = \frac{\rho_0}{\theta_{0z}}\left\{-\frac{f^2\theta_{0z}(1-\kappa)}{\rho_0 R\Pi_0\hat{\theta}_0}\psi'_B(k) - \right.$$

$$\frac{f^2\kappa}{\rho_0\hat{\Pi}_{0z}}\left(\theta_{0z} + \frac{2g\Pi_{0zz}}{c_p\hat{\Pi}_{0z}^2}\right)\left(\frac{\alpha_1(k)}{r^p(k+1)-r^p(k)}\left[\frac{\Pi_0(k+1)\rho_0(k+1)}{p_0(k+1)}\psi'_B(k+1) - \frac{\Pi_0(k)\rho_0(k)}{p_0(k)}\psi'_B(k)\right] + \right.$$

$$\left.\frac{\beta_1(k)}{r^p(k)-r^p(k-1)}\left[\frac{\Pi_0(k)\rho_0(k)}{p_0(k)}\psi'_B(k) - \frac{\Pi_0(k-1)\rho_0(k-1)}{p_0(k-1)}\psi'_B(k-1)\right]\right) +$$

$$\frac{f^2 g\kappa}{\rho_0 c_p\hat{\Pi}_{0z}^2}\frac{1}{r^\theta(k)-r^\theta(k-1)}\left(\frac{\Pi_0(k+1)\rho_0(k+1)\psi'_B(k+1)/p_0(k+1) - \Pi_0(k)\rho_0(k)\psi'_B(k)/p_0(k)}{r^p(k+1)-r^p(k)} - \right.$$

$$\left.\left.\frac{\Pi_0(k)\rho_0(k)\psi'_B(k)/p_0(k) - \Pi_0(k-1)\rho_0(k-1)\psi'_B(k-1)/p_0(k-1)}{r^p(k)-r^p(k-1)}\right)\right\}. \qquad (15)$$

5.2 The unbalanced system

$$\mathbf{A}_z p'_u = -\frac{f^2(1-\kappa)}{R\Pi_0\hat{\theta}_0}p'_u(k) -$$

$$\frac{f^2\kappa\rho_0}{\hat{\Pi}_{0z}}\left(1 + \frac{2g\Pi_{0zz}}{c_p\theta_{0z}\hat{\Pi}_{0z}^2}\right)\left(\frac{\alpha_1(k)}{r^p(k+1)-r^p(k)}\left[\frac{\Pi_0(k+1)}{p_0(k+1)}p'_u(k+1) - \frac{\Pi_0(k)}{p_0(k)}p'_u(k)\right] - \right.$$

$$\frac{\beta_1(k)}{r^p(k)-r^p(k-1)}\left[\frac{\Pi_0(k)}{p_0(k)}p'_u(k)-\frac{\Pi_0(k-1)}{p_0(k-1)}p'_u(k-1)\right]\Bigg)+$$
$$\frac{f^2 g\kappa\rho_0}{c_p\theta_{0z}\hat{\Pi}_{0z}^2}\frac{1}{(r^\theta(k)-r^\theta(k-1))}\left(\frac{\Pi_0(k+1)p'_u(k+1)/p_0(k+1)-\Pi_0(k)p'_u(k)/p_0(k)}{r^p(k+1)-r^p(k)}-\right.$$
$$\left.\frac{\Pi_0(k)p'_u(k)/p_0(k)-\Pi_0(k-1)p'_u(k-1)/p_0(k-1)}{r^p(k)-r^p(k-1)}\right). \qquad (16)$$

6. Are there any special conditions on the source term needed to solve the Screened Poisson Equation?

The ordinary Poisson equation formulated on the sphere requires that the source term has a global mean of zero. Are there any analogous requirements for the screened Poisson equation? We will look at this problem in one dimension.

The one-dimensional screened Poisson equation has the generic form [4],

$$\left(\nabla_z^2 - \kappa^2\right)\psi(x) = \rho(x). \qquad (17)$$

In the problems considered here, κ^2 is a constant. Write in terms of Fourier transforms (overbars),

$$\left(\nabla_z^2 - \kappa^2\right)\int \mathrm{d}k'\,\bar{\psi}(k')\exp(ik'x) = \int \mathrm{d}k'\,\bar{\rho}(k')\exp(ik'x).$$

Multiply each side by $\exp(-ikx)$ and integrate over the globe.

$$-\int \mathrm{d}x\int \mathrm{d}k'\left(k'^2+\kappa^2\right)\bar{\psi}(k')\exp(i[k'-k]x) = \int \mathrm{d}x\int \mathrm{d}k'\,\bar{\rho}(k')\exp(i[k'-k]x),$$

$$\left(k^2+\kappa^2\right)\bar{\psi}(k) = \bar{\rho}(k) \qquad (18)$$

Thus as long as $\kappa^2 > 0$ (ie non-zero), this equation in invertible for all k, and so there are no conditions needed on the source term.

7. References

[1] Press W.H., Teukolsky S.A., Vetterling W.T., Flannery B., Numerical Recipes.

[2] Zerroukat M., Iterative solvers, Met Office document (October 2000).

[3] Bannister R.N., Approximate 'vertical-only' preconditioning of the PV equations (2008) - download via PV project web page (`http://www.met.rdg.ac.uk/~ross/DARC/PVcv/Tridiag.pdf`).

[4] `http://en.wikipedia.org/wiki/Screened_Poisson_equation`