Density scaling

Ross Bannister, May 5th/May 11th 2006

Aim: To develop a PV operator that is a function of a new variable that is less affected by density scaling than in the present formula.

The present formula

At the moment, PV is a function of streamfunction, ψ, in the following way:

$$PV = \frac{\alpha}{\rho_0}\nabla^2\psi - \frac{\beta}{\rho_0^2}\varepsilon p + \frac{\gamma}{\rho_0}\frac{\partial}{\partial z}(\eta p) + \frac{\kappa}{\rho_0}\frac{\partial^2}{\partial z^2}(\eta p). \quad (1)$$

PV, ψ and p are incremental quantities, and α, β, ε, γ, κ and η are shorthand for groups of reference state quantities. ρ_0 is reference density, which we assume is a function of z only. The linear balance equation, relating ψ and p is denoted by,

$$\begin{aligned} p &= \rho_0\nabla^{-2}\nabla\cdot(f\nabla\psi), \\ &= \rho_0\mathbf{L}\psi. \quad (2) \end{aligned}$$

Combining (1) and (2),

$$PV = \frac{\alpha}{\rho_0}\nabla^2\psi - \frac{\beta}{\rho_0^2}\varepsilon\rho_0\mathbf{L}\psi + \frac{\gamma}{\rho_0}\frac{\partial}{\partial z}(\eta\rho_0\mathbf{L}\psi) + \frac{\kappa}{\rho_0}\frac{\partial^2}{\partial z^2}(\eta\rho_0\mathbf{L}\psi). \quad (3)$$

This operator is strongly modulated by density.

Change of variable

What change of dependent variable will lessen the density dependence? Let a new variable be s. This is what will be solved for in the GCR.

$$\text{Try 1}: \psi = \rho_0 s, \quad (4a)$$

$$\text{Try 2}: \psi = s/\rho_0, \quad (4b)$$

$$\text{Try 3}: \psi = \rho_0^2 s. \quad (4c)$$

Tries 1 and 2 are documented in the sections below, but all have been tested. It seems to me that try 2 is like Mike Sharpe's method.

Try 1

Put (4a) into (3),

$$\begin{aligned} PV &= \frac{\alpha}{\rho_0}\nabla^2\rho_0 s - \frac{\beta}{\rho_0^2}\varepsilon\rho_0\mathbf{L}(\rho_0 s) + \frac{\gamma}{\rho_0}\frac{\partial}{\partial z}(\eta\rho_0\mathbf{L}(\rho_0 s)) + \frac{\kappa}{\rho_0}\frac{\partial^2}{\partial z^2}(\eta\rho_0\mathbf{L}(\rho_0 s)), \\ &= \alpha\nabla^2 s - \beta\varepsilon\mathbf{L}s + \quad (\textit{terms } 1 \,\&\, 2) \\ &\quad \frac{\gamma\eta}{\rho_0}\mathbf{L}s\frac{\partial\rho_0^2}{\partial z} + \gamma\rho_0\frac{\partial(\eta\mathbf{L}s)}{\partial z} + \quad (\textit{term } 3) \\ &\quad \frac{\kappa}{\rho_0}\frac{\partial}{\partial z}\left(\eta\mathbf{L}s\frac{\partial\rho_0^2}{\partial z} + \rho_0^2\frac{\partial(\eta\mathbf{L}s)}{\partial z}\right), \quad (\textit{term } 4) \\ &= \alpha\nabla^2 s - \beta\varepsilon\mathbf{L}s + \quad (\textit{terms } 1 \,\&\, 2) \end{aligned}$$

$$\frac{\gamma\eta}{\rho_0}\mathbf{L}s\frac{\partial\rho_0^2}{\partial z} + \gamma\rho_0\frac{\partial(\eta\mathbf{L}s)}{\partial z} + \qquad (term\ 3)$$

$$\frac{\kappa}{\rho_0}\eta\mathbf{L}s\frac{\partial^2\rho_0^2}{\partial z^2} + \frac{\kappa}{\rho_0}\frac{\partial(\eta\mathbf{L}s)}{\partial z}\frac{\partial\rho_0^2}{\partial z} + \kappa\rho_0\frac{\partial^2\eta\mathbf{L}s}{\partial z^2} + \frac{\kappa}{\rho_0}\frac{\partial\rho_0^2}{\partial z}\frac{\partial\eta\mathbf{L}s}{\partial z}. \qquad (term\ 4)$$

We have not removed density scaling from the problem, but we have gained two terms that are not affected by density scaling (terms 1 and 2).

Try 2

Put (4b) into (3),

$$PV = \frac{\alpha}{\rho_0}\nabla^2 s/\rho_0 - \frac{\beta}{\rho_0^2}\varepsilon\rho_0\mathbf{L}s/\rho_0 + \frac{\gamma}{\rho_0}\frac{\partial}{\partial z}(\eta\rho_0\mathbf{L}s/\rho_0) + \frac{\kappa}{\rho_0}\frac{\partial^2}{\partial z^2}(\eta\rho_0\mathbf{L}s/\rho_0),$$

$$= \frac{\alpha}{\rho_0^2}\nabla^2 s - \frac{\beta}{\rho_0^2}\varepsilon\mathbf{L}s + \frac{\gamma}{\rho_0}\frac{\partial}{\partial z}(\eta\mathbf{L}s) + \frac{\kappa}{\rho_0}\frac{\partial^2}{\partial z^2}(\eta\mathbf{L}s).$$

This has not removed density from the problem, but it has removed the need to operate with vertical derivatives on density.

Tests

A quick way of testing out try 1 and try 2 is as follows. The tests done here can be generalised as $\psi = s\rho_0^n$. We try $n = 0$ (no transform), $n = 1$ (try 1) and $n = -1$ (try 2), etc.

Set up initial streamfunction $\psi_{(0)}$

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Compute initial modified streamfunction $s_{(0)} = \psi_{(0)}\rho_0^{-n}$

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Run GCR with s as variable

(for iteration k, calculations of PV first make transformation $\psi_{(k)} = s_{(k)}\rho_0^n$

↓

Solution in terms of s is $s_{(K)}$

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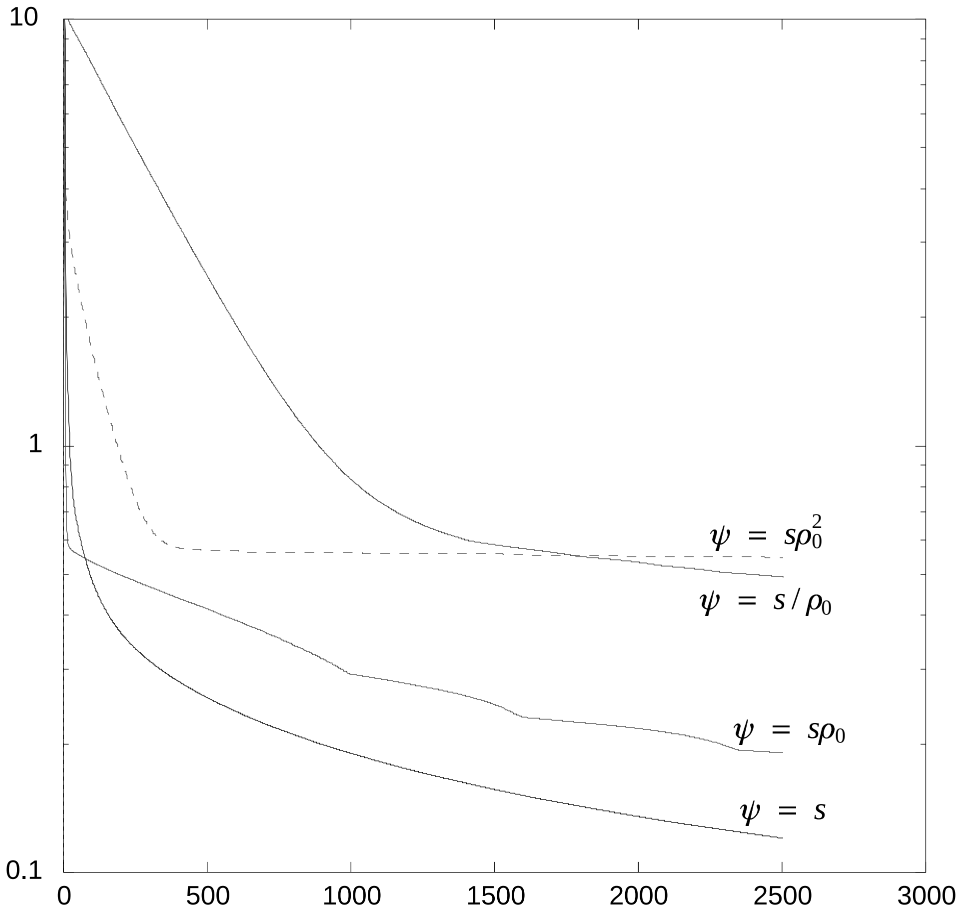
Solution in terms of ψ is $\psi_{(K)} = s_{(K)}\rho_0^n$



This plot shows the error $\int \mathrm{d}V\,(PV_{RHS} - PV(s))/PV_{RHS}$ versus iteration (an error of 1 means that the error is of order of the RHS PV on average). No test was able to 'beat' the untransformed case. No preconditioning was used in any of these tests, and the polar point has no off-diagonal terms in the horizontal in the PV operator.