Adjoint coding

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1. When the forward calculation is a single summation

Forward calculation

INPUTS: p_k; OUTPUT: L

$$L = \sum_{k=1}^{N} A_k p_k .$$

Here there is <u>one</u> summation and <u>one</u> output.

Adjoint calculation

INPUT: $\hat{L}$; OUTPUTS: $\hat{p}_k$

$$\frac{\partial}{\partial p_k} = \frac{\partial L}{\partial p_k}\frac{\partial}{\partial L},$$

$$\hat{p}_k = A_k \hat{L}.$$

Now there are <u>no</u> summations and <u>many</u> outputs (i.e. different k).

2. When the forward calculation is a double summation

Forward calculation

INPUTS: p_k; OUTPUT: L

$$L = \sum_{k=1}^{N} A_k I_k, \qquad \text{where} \qquad I_k = \sum_{\substack{k'=1 \\ (k' \leqslant k)}}^{k} B_{k'} p_{k'} .$$

Here there are <u>two</u> summations and <u>one</u> output.

Adjoint calculation

INPUT: $\hat{L}$; OUTPUTS: $\hat{p}_k$

$$\frac{\partial}{\partial p_{k'}} = \sum_{\substack{k=k' \\ k' \leqslant k}}^{N} \frac{\partial I_k}{\partial p_{k'}}\frac{\partial}{\partial I_k}, \qquad \frac{\partial}{\partial I_k} = \frac{\partial L}{\partial I_k}\frac{\partial}{\partial L},$$

$$\hat{p}_{k'} = \sum_{\substack{k=k' \\ k' \leqslant k}}^{N} B_{k'} \hat{I}_k, \qquad \hat{I}_k = A_k \hat{L}.$$

Now there is <u>one</u> summation and <u>many</u> outputs (i.e. different k'). It is important to distinguish between k and k'. The above can be written as,

$$\hat{p}_{k'} = \sum_{\substack{k=k' \\ k' \leqslant k}}^{N} B_{k'} A_k \hat{L} = \hat{L} B_{k'} \sum_{\substack{k=k' \\ k' \leqslant k}}^{N} A_k .$$

This result is found to pass the adjoint test,

$$\langle O\vec{v},\ O\vec{v}\rangle = \langle O^T O\vec{v},\ \vec{v}\rangle .$$