THE UNIVERSITY OF READING

The Quaternionic Structure of the Equations of Geophysical Fluid Dynamics

Jonathan Matthews

A thesis submitted for the degree of Doctor of Philosophy
School of Mathematics, Meteorology and Physics
September 2006
I confirm that this is my own work and the use of all material from other sources has been properly and fully acknowledged.

Jonathan Matthews
Abstract

A new mathematical theory is derived for a general three-dimensional fluid flow in the framework of quaternionic algebra.

This quaternionic formulation is derived in two separate ways: firstly, by a 4-vector representation of the growth and rotation rates of the vorticity and, secondly, by a 4-vector representation of the vorticity.

This general theory does not assume that the vorticity takes any particular form. However, certain constraints and limitations to the theory are discussed. The corresponding general complex structure for this problem is also derived.

This general theory is explained within the context of a hierarchy of fluid dynamical models starting with the incompressible, three-dimensional Euler equations. The growth and rotation variables are discussed in the context of this model and the evolution of the vortex stretching, which leads to the introduction of the pressure Hessian operator, is illustrated as vital in “closing” this problem. The limitations, constraints and in some cases, breakdown of the quaternionic structure is seen when further fluid dynamic models that include, but is not restricted to, the Navier-Stokes, shallow-water and hydrostatic, primitive equations are discussed.

Finally, in the appendix, the growth and rotation rates are computed from data produced by the UK Meteorological Office’s Unified Model and some basic subjective analysis is carried out by comparing the result for the leading order stretching rate with corresponding diagnostics.
Acknowledgements

I may only get to do this once so I’m really going to go for it. Thanks, of course, to my supervisors, Alan O’Neill, John Gibbon, William Lahoz but most of all to Ian Roulstone who really got me interested in this problem and who was always available for advice or just a chat. Thanks go to the members of my thesis committee John Thuburn and especially Brian Hoskins, who managed, even in the most serious of discussions, to mention my starring roles in the departmental pantomime. Thanks go to my fellow Ph.D. students, but especially to Andy, Tom, Dan and Leon, I hope I never have to live with any of you again! Thanks also to my Met Office supervisor Mike Cullen and the help that I received during my time in Exeter from Tim Payne and Sean Milton. A special mention to Darryl Holm; I will always remember the time you woke me up at 8 a.m. on a Saturday while at a conference in Potsdam to discuss my work; with hindsight please accept my appreciation.

Thanks of course to my parents, Sharon and John, who probably won’t read any of this apart from this page, but thank-you with all my heart for everything that you have done. To my grandparents, Gladys and Ernie & Daphne and Jack, who have always been there for me and to my brother, Alexander; I’m sure a reason for including you will come to me.

Special thanks to my dearest friend Mark, nothing more needs to be said and to my friends across the pond, especially Bonnie, Joe and Marilyn, love you!

Finally I have two people to thank, without them I would not be writing this today. One is Frank Berkshire (the only other UK MLB fan) and the second is the late Nick Real, who died only a few weeks ago. I would loved to have shown you this and everything else that I have achieved. I’m sure I will one day.
# Contents

1 Introduction 1
   1.1 Background and motivation 1
   1.2 The role of quaternions 4
   1.3 Approximations to Euler & balanced models 6
   1.4 Questions addressed in this thesis 7
   1.5 Thesis plan 7

2 Quaternionic structure of a general 3D vorticity equation 9
   2.1 Theory 9
   2.2 Mathematical derivation 10
      2.2.1 Evolution equations for the stretching rate and the alignment vector 10
      2.2.2 Quaternions and their corresponding algebra 16
      2.2.3 An inherent/basic quaternionic structure 18
   2.3 Alternative derivation of the quaternionic structure 20
   2.4 Corresponding complex structure 21
   2.5 Mathematical constraints 22
   2.6 Conditions and limitations of the theory 23
   2.7 Summary 24

3 The inertial incompressible Euler equations 26
   3.1 Equations of motion 27
   3.2 The stretching rate and vorticity alignment 29
      3.2.1 The role of the local angle $\phi$ 30
   3.3 Equivalent condition for potential singular solutions 31
   3.4 The evolution of the vorticity stretching rate and pressure Hessian 33
<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.5</td>
<td>The quaternionic structure in the incompressible Euler equations</td>
<td>35</td>
</tr>
<tr>
<td>3.5.1</td>
<td>Burgers’ solutions to the Euler equations</td>
<td>37</td>
</tr>
<tr>
<td>3.6</td>
<td>The complex structure in the incompressible Euler equations</td>
<td>38</td>
</tr>
<tr>
<td>3.7</td>
<td>The constraint equation for the Euler equations</td>
<td>39</td>
</tr>
<tr>
<td>3.8</td>
<td>The work of Adler &amp; Moser and the complex Schrödinger equation</td>
<td>40</td>
</tr>
<tr>
<td>3.9</td>
<td>Equivalent conditions for potential singular solutions II</td>
<td>43</td>
</tr>
<tr>
<td>3.10</td>
<td>Quaternionic form of the momentum equations</td>
<td>44</td>
</tr>
<tr>
<td>3.11</td>
<td>Evolution equations for the pressure Hessian variables</td>
<td>47</td>
</tr>
<tr>
<td>3.11.1</td>
<td>A quaternionic representation of the pressure 4-vector $q_p$</td>
<td>49</td>
</tr>
<tr>
<td>3.12</td>
<td>Comparison analysis with the Navier-Stokes equations</td>
<td>49</td>
</tr>
<tr>
<td>3.12.1</td>
<td>The classical approach</td>
<td>50</td>
</tr>
<tr>
<td>3.12.2</td>
<td>Thesis approach</td>
<td>51</td>
</tr>
<tr>
<td>3.13</td>
<td>Summary</td>
<td>53</td>
</tr>
<tr>
<td>4</td>
<td>The Euler equations with rotation</td>
<td>55</td>
</tr>
<tr>
<td>4.1</td>
<td>Equations of motion</td>
<td>55</td>
</tr>
<tr>
<td>4.2</td>
<td>The vorticity equation</td>
<td>57</td>
</tr>
<tr>
<td>4.3</td>
<td>Constant density fluid</td>
<td>59</td>
</tr>
<tr>
<td>4.3.1</td>
<td>The quaternionic formulations of the equation dependent term</td>
<td>61</td>
</tr>
<tr>
<td>4.3.2</td>
<td>The Ohkitani result in 4-vector form</td>
<td>62</td>
</tr>
<tr>
<td>4.3.3</td>
<td>Brief mention of the corresponding complex structure</td>
<td>64</td>
</tr>
<tr>
<td>4.3.4</td>
<td>The corresponding constraint equation</td>
<td>64</td>
</tr>
<tr>
<td>4.3.5</td>
<td>Beale-Kato-Majda calculation for the Euler equations with rotation</td>
<td>65</td>
</tr>
<tr>
<td>4.4</td>
<td>Analysis for a barotropic fluid</td>
<td>67</td>
</tr>
<tr>
<td>4.4.1</td>
<td>Incompressible case</td>
<td>68</td>
</tr>
<tr>
<td>4.4.2</td>
<td>Constraint equation for a barotropic flow</td>
<td>70</td>
</tr>
<tr>
<td>4.5</td>
<td>Summary</td>
<td>70</td>
</tr>
</tbody>
</table>
5 The **breakdown of the hydrostatic case**  
5.1 Momentum and vorticity equations ........................................ 72  
5.2 The non-dimensional momentum and continuity equations ........ 76  
5.3 The non-dimensional vorticity equation .................................. 80  
5.4 Hydrostatic balance for a constant density and barotropic fluid ... 82  
5.4.1 Constant density case .................................................... 82  
5.4.2 The barotropic case ..................................................... 86  
5.5 The two-dimensional quasi-geostrophic thermal active scalar ...... 90  
5.6 Summary ................................................................. 94  

6 The **non-hydrostatic and hydrostatic, primitive equations** ......... 96  
6.1 Equations of motion ....................................................... 96  
6.1.1 The quaternionic form of the primitive equations ............... 99  
6.2 Non-dimensional form of the primitive equations .................. 99  
6.3 A closer consideration of the evolution equation for the stretching  
rate .................................................................................. 105  
6.4 Summary ................................................................. 106  

7 Conclusions ....................................................................... 108  

8 **Appendix - numerical treatment of the vortex stretching and rotation**  
variables ............................................................................. 111  
8.1 The momentum and vorticity equation in spherical polar co-ordinates111  
8.2 The grid structure .............................................................. 114  
8.2.1 The co-ordinate system .................................................. 114  
8.2.2 Grid Spacing and variable placement ............................... 115  
8.3 Discretization of model variables ........................................ 116  
8.4 UM model data and grid spacing ....................................... 118  
8.5 Numerical consideration of the different vortex variables ...... 118  
8.5.1 Numerical representation of the vorticity components ...... 118  
8.5.2 The vortex stretching components .................................. 119
8.5.3 The stretching rate and negative horizontal divergence . . . 120
8.5.4 The components of the vortex alignment variable . . . . . . 122
8.6 The numerical analysis of the development of singular solutions . . 122
8.7 Summary . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 126

9 Glossary 128

9.1 Glossary of mathematical symbols . . . . . . . . . . . . . 128
  9.1.1 Chapter 2 . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 128
  9.1.2 Chapter 3 . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 129
  9.1.3 Chapter 4 . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 129
  9.1.4 Chapter 5 . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 130
  9.1.5 Chapter 6 . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 130
  9.1.6 Chapter 7 . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 130
9.2 Vector and scalar laws . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 130
9.3 Integral theorems . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 131
9.4 Spherical-polar form of vector operators . . . . . . . . . . . . . . . . . . 131

References 133
List of Figures

2.1 Vortex line with tangent vorticity vector $\omega$. The vectors $\{\omega, \chi, \omega \times \chi\}$ form an ortho-normal co-ordinate system and the vectors $\omega, \sigma, \omega \times \chi$ are co-planar. .......................................................... 13

8.1 The unit vectors $\mathbf{I}, \mathbf{J}, \mathbf{K}$ associated with the directions $Ox, Oy, Oz$ in the rotated system and the unit vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$ associated with the zonal, meridional and radial directions at a point $P$ having longitude $\lambda$ and latitude $\phi$ in the related system .................................................. 112

8.2 Arakawa C-grid showing staggered $u$ and $v$ at $(I \pm 1/2, K \pm 1/2)$ and $(I \pm 1/2, J, K \pm 1/2)$ respectively. The relative position of these variables along with the corresponding vorticity is shown ............................. 116

8.3 Charney-Philips grid staggering. The $\theta$ and $\rho$-levels correspond to the integral value $K$ and half-integral values $K \pm 1/2$ respectively. The height of $\eta$ level is shown as the sum of the three parts, $r(E)$ - the mean radius of the Earth, $r(O)$ - the height due to orography and $r(\rho, \theta)$ the height at a particular $(\rho, \theta)$ level ................................................. 117

8.4 The first component of the vorticity vector $\omega_\lambda$ at a height level of approximately 0.1km above the orography .................................................. 119

8.5 The second component of the vorticity vector $\omega_\phi$ ................................................. 119

8.6 The third component of the vorticity vector $\omega_r$ ................................................. 120

8.7 The first component of the vortex stretching term $\sigma_\lambda$ ................................................. 121

8.8 The second component of the vortex stretching term $\sigma_\phi$ ................................................. 121

8.9 The third component of the vortex stretching term $\sigma_r$ ................................................. 122

8.10 The stretching rate $\alpha$ ................................................. 123

8.11 The negative horizontal divergence field $-\nabla \cdot \mathbf{v}$ ................................................. 123

8.12 The first component of the vortex rotation vector $\chi_\lambda$ ................................................. 124
<table>
<thead>
<tr>
<th>Figure</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>8.13</td>
<td>The second component of the vortex rotation vector $\chi_\phi$</td>
<td>124</td>
</tr>
<tr>
<td>8.14</td>
<td>The third component of the vortex rotation vector $\chi_r$</td>
<td>125</td>
</tr>
<tr>
<td>8.15</td>
<td>The $X$ variable given by $X^2 = \alpha^2 + \chi \cdot \chi$</td>
<td>125</td>
</tr>
<tr>
<td>8.16</td>
<td>The maximum row sum of the matrix $(P' + \Omega^*)$</td>
<td>126</td>
</tr>
</tbody>
</table>
Chapter 1

Introduction

1.1 Background and motivation

Why do we study applied mathematics or more specifically fluid dynamics? Apart from our curiosity to understand the world’s natural phenomena and the corresponding benefits that this can bring to industry, business and commerce, there can be the additional unexpected but equally desirable benefits of fame and money. In 2000, the Clay Mathematics Institute of Cambridge, Massachusetts (CMI) named seven classical research questions that have, so far, remained unsolved. There is a $7 million prize fund ($1 million per problem) for the solutions to these problems. Of these seven, one relates to the Navier-Stokes equations. As described in the official problem description Fefferman (2006):

The Euler and Navier-Stokes equations describe the motion of a fluid in $\mathbb{R}^n$ ($n = 2$ or $3$). These equations are to be solved for an unknown velocity vector $u(x, t) = (u_i(x, t))_{1 \leq i \leq n} \in \mathbb{R}^n$ and pressure $p(x, t) \in \mathbb{R}$, defined for position $x \in \mathbb{R}^n$ and time $t \geq 0$. We restrict attention here to incompressible fluids filling all of $\mathbb{R}^n$. The Navier-Stokes equations are given by
\[
\frac{\partial u_i}{\partial t} + \sum_{j=1}^{n} u_j \frac{\partial u_i}{\partial x_j} = \nu \Delta u_i - \frac{\partial p}{\partial x_i} + f_i(x,t), \quad (1.1)
\]

\[
\text{div} u = \sum_{i=1}^{n} \frac{\partial u_i}{\partial x_i} = 0, \quad (1.2)
\]

with initial conditions

\[
u(x,0) = u^0(x) \quad (x \in \mathbb{R}^n). \quad (1.3)
\]

Here, \(u^0(x)\) is a given \(C^\infty\) divergence-free vector field on \(\mathbb{R}^n\), \(f_i(x,t)\) are the components of a given, externally applied force, \(\nu\) is a positive coefficient (the viscosity), and \(\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}\) is the Laplacian in the space variables. The Euler equations are (1.1), (1.2) and (1.3) with \(\nu\) set equal to zero.

Although in no way does this thesis attempt to solve this problem it hopefully adds to our understanding of this complex set of nonlinear, partial differential equations for which, to date, very little is known. As an actual solution defined by the Clay Institute is a long way in the future we restrict our focus within the field of fluid dynamics to important (and in some senses, equally important) unresolved research problems that will hopefully have an impact in our understanding of fluid flows in general. One such famous, unanswered question is:

\textit{Are there smooth solutions with finite energy of the three dimensional Euler equations that develop singularities in finite time?}

The search for singular solutions to the Euler equations is a much studied research topic because it involves the study of nonlinear intensification of vorticity as well as the creation of small scales (at high Reynolds numbers) in turbulent fluid flows. This problem has been explored in terms of the growth of vorticity and the way in which the vorticity stretches and compresses. In fact, one key result in the study of singularities in three-dimensional Euler flow is the well known theorem stated in Beale \textit{et al.} (1984). Although this theorem does not say when a singu-
larity will occur it does say that no quantity within the flow will blow-up (become infinite) in finite time \( t \to t^* \) without the quantity

\[
\int_0^t \| \omega(\tau) \|_{\infty} \, d\tau \to \infty,
\]

where the vorticity

\[ \omega = \text{curl } u, \]

and \( t \to t^* \). It has been further shown in Constantin et al. (1996), that the \( L^\infty \)-norm of the vorticity seen in (1.4) can be reduced to a \( L^q \)-norm for some finite \( q \) provided that certain constraints are applied to the direction of vorticity.

Three-dimensional Euler vorticity growth is driven by the vector \((\omega \cdot \nabla) u\). This vector plays a fundamental role in determining whether or not a singularity forms in finite time. Major computational studies in this direction can be found in Kerr (1993, 2005); Brachet et al. (1983, 1992); Pumir and Siggia (1990) and Pelz (2001).

Furthermore, singularities will not develop in the solutions to three-dimensional incompressible Euler flow if the direction of vorticity is smooth (and of course the velocity remains finite) Constantin et al. (1996) or if the angle between local vortex lines does not become too large Constantin (1994). Further studies into the direction of vorticity have been considered in Cordoba and Fefferman (2001), Deng et al. (2005, 2006) and Chae (2005, 2006). However, what is not fully understood is what governs the direction and how certain quantities, such as the vorticity, orientate within the flow. Also with respect to turbulent fluid flows, it is thought that the stretching and direction of the vorticity in the three-dimensional Euler equations may obey certain, unknown, geometric properties.

In previous years, advancements have been made in our general understanding of vorticity and specifically its stretching, compression and alignment by considering the local angle that lies between the vorticity \( \omega \) and \( S\omega \), where \( S \) is the rate-of-strain or deformation matrix given by the symmetric part of the velocity gradient matrix \( \nabla u \). By transforming the corresponding three-dimensional Euler vorticity
equation and considering the evolution equation for the unit vector of the vorticity, two new variables can be defined, and they are

\[\alpha = \frac{\omega \cdot S\omega}{\omega \cdot \omega}, \quad \chi = \frac{\omega \times S\omega}{\omega \cdot \omega}.\] (1.6)

The first is a scalar known as the stretching rate (Constantin (1994)), which relates only to stretching and compression of vorticity while the second vector term in (1.6) is the spin rate and provides information regarding the direction and alignment of the vorticity in terms of its orientation with \(S\omega\); these two variables were first introduced in Galanti et al. (1997). Furthermore, in Gibbon et al. (2000) and later in Gibbon (2002) the corresponding evolution equations for (1.6) were derived in terms of two similar variables \((\alpha_p, \chi_p)\). These new variables differ from the original ones by replacing the strain matrix with the pressure Hessian matrix. Literature on the pressure Hessian \(P\) and its interplay with the strain matrix \(S\) has appeared in Galanti et al. (1997), Majda and Bertozzi (2001) and Chae (2006).

### 1.2 The role of quaternions

In Gibbon (2002) it was first noted that the two differential equations for the stretching rate \(\alpha\) and the alignment vector \(\chi\) are given by

\[
\frac{D\alpha}{Dt} + \alpha^2 - |\chi|^2 = -\alpha_p, \\
\frac{D\chi}{Dt} + 2\chi\alpha = -\chi_p, \\
\] (1.7)

where each term is explained in Chapter 3. These equations can be re-written as a single evolution equation in terms of a single 4-vector \(q = (\alpha, \chi)^T\) and corresponding 4-vector \(q_p = (\alpha_p, \chi_p)^T\). This single evolution equation is in fact a quaternionic Riccati equation.

Quaternions, first described in 1843 by Sir William Rowan Hamilton, are a non-commutative extension in \(\mathbb{R}^4\) of complex numbers. Hamilton was looking for a way of extending complex numbers to higher spatial dimensions. Although he
couldn’t achieve this in three dimensions, in four dimensions he derived quater-
nions. In the previous century and a half (until only recently), quaternions have
fallen in and out of fashion (Tait (1890)), and were generally proved to be unpopular
compared to vector-based notation (even though, for example, early formulations
of Maxwell’s equations used a quaternion based notation) as they require 3-vector
algebra to work them. However, in recent years, quaternions have had something of
a revival and their algebra and structure have been exploited in the field of computer
graphics to represent rotations and the orientation of objects in three-dimensional
space (Hanson (2006) and Kuipers (1999)). This is not altogether surprising as
the general representation of a quaternion can be expressed in terms of the com-
plex Pauli matrices, which in theoretical physics are known to represent rotations.
The reason why quaternions are used to represent rotations/orientations is because
their form is smaller in size than other common representations such as matrices,
and combining many quaternionic transformations is more numerically stable than
combining a large number of matrix transformations. They have also been used in
such research areas as signal processing, orbital mechanics and control theory.

At this junction it may be beneficial to give a taster, with no theo-
rematical justification (that will come later) of the direct role that quaternions
play in this research. Quaternions form an algebra in $\mathbb{R}^4$ and for ease of notation
can be represented as column vectors in the form $q_i = (\alpha_i, \chi_i)^T$. What is meant
specifically by the phrase “forms an algebra in $\mathbb{R}^4$” is that there must be some
means of adding and multiplying two different quaternions together. Addition is
simple and involves the adding of corresponding column entries. The multiplication
operator of two quaternions generates a linear vector space and is given by the
following direct product

$$q_1 \otimes q_2 = \begin{pmatrix}
\alpha_1 \alpha_2 - \chi_1 \cdot \chi_2 \\
\alpha_1 \chi_2 + \alpha_2 \chi_1 + \chi_1 \times \chi_2
\end{pmatrix};$$

(1.8)

the justification for this will be seen in the next chapter. It is this form in the ex-
pression of the quaternionic (multiplication) algebra that corresponds directly to the
algebra of the fluid dynamics for the vorticity stretching $\alpha$ and the alignment vector $\chi$.

Quaternions play an important role in the theory and study of manifolds in four dimensional space, and from this study it has been shown that the physics of particles and fields are governed by certain geometric properties. The key underlying belief behind the earlier research of Gibbon (2002) is that a natural quaternionic structure points to a corresponding geometric structure within the original nonlinear, partial differential equations governing the fluid flow, about which currently little is known. However, one problem with the quaternionic relationship in the three-dimensional incompressible, Euler equations given in Gibbon (2002) is that the structure is in the dependent vorticity variables and for a full quaternionic formulation the independent spatial variables would also have to be written in quaternionic form. In Gibbon (2002), the quaternionic Riccati equation can be transformed quite simply to a complex zero-eigenvalue Schrödinger equation whose potential is based on the $\alpha_p$ and $\chi_p$ variables seen in (1.7). An infinite set of solutions to the scalar zero-eigenvalue Schrödinger equation has been discussed in Adler and Moser (1978) and whose solutions are transformed to the current problem in Gibbon (2002).

1.3 Approximations to Euler & balanced models

It has hopefully been made quite clear in the first part of the introduction that very little in the abstract mathematical sense is known about the Euler equations in $\mathbb{R}^4$. One way of trying to overcome this is to transform the original momentum and mass conservation equations into a form that is more susceptible to mathematical analysis, which leaves the underlying structure and scope of the equations intact. This has already been touched upon by considering transformations of the vorticity equation.

However, a second way of progressing is by making suitable approximations, based on mathematical and numerical analysis, to the original or parent dynamics.
For example, in the study of weather forecasting and climatology, numerical models are based on the hydrostatic, primitive equations which are, in essence, an approximate form of the Euler equations generalized to include rotation on a spherical planet. One particular subset of approximate equations is known as balanced models - these models are constructed to remove or eliminate the “inertia-gravity” waves that can occur in numerical weather prediction or in the primitive equation models. It has been shown in Roubtsov and Roulstone (1997, 2001) that there exists, in the co-ordinate transformations equations, a quaternionic structure in a particular set of balanced models. These two quaternionic structures, in Euler and the balanced models, have certain similarities such as the major role that the pressure Hessian matrix plays in both analyses, but conversely, there are certain key differences such as the quaternionic structure in Euler is based on the vorticity variables while the structure in the balanced models is not.

### 1.4 Questions addressed in this thesis

As balanced models are based on the Euler equations it would be advantageous to know how certain structures are inherited from the parent dynamics so as to give a greater understanding mathematically of these balanced models, their solutions and approximations in general.

The main aim of this thesis can be summed up as giving a comprehensive understanding of the role the vorticity variables play, both in Euler, and its approximations, in particular answering the key question:

What form does the quaternionic structure take (if any) as each successive approximation to the three-dimensional incompressible Euler equations is made?

### 1.5 Thesis plan

In Chapter 2 the question regarding the form that the quaternionic structure takes in successive approximations is addressed and a new quaternionic structure for a gen-
eral, (equation-independent) vorticity equation is derived with certain assumptions and conditions.

In Chapter 3 a review of the development of the quaternionic formulation for the three-dimensional incompressible Euler equations (seen in Gibbon (2002) and earlier papers) is first given, and is then further expanded upon and placed within a general framework of the results described in Chapter 2.

In Chapter 4 the earlier work is considered in a rotating reference frame. This is done to enable further mathematical and theoretical approximations to be made to this set of equations later. The particular cases of constant density and barotropic flows are discussed.

In Chapters 5 and 6 the particular case of the hydrostatic balance equations and the associated breakdown of the quaternionic structure is discussed. This breakdown is then resolved in terms of the hydrostatic, primitive equations. The form that the stretching rate and alignment vector variables take, along with their corresponding evolution equations, is considered.

In the Appendix the \((\alpha, \chi)\) variables are diagnosed using data from the UK Meteorological Office’s Unified Model, and certain key results presented in earlier chapters are considered along with this data both numerically and graphically.
Chapter 2

Quaternionic structure of a general 3D vorticity equation

In the Introduction the fundamental question raised was “how is the quaternionic structure of the incompressible (non-rotating) Euler equations affected by subsequent approximations from this same set of parent/governing equations?” Instead of considering this rather explicit question regarding the Euler equations, a more general theory concerning a quaternionic structure for a given three-dimensional vorticity equation is developed. Later, in Chapter 3, specific examples in the context of this general theory will be considered, starting with the incompressible Euler equations (Gibbon (2002)), and then developing the problem further to consider the rotational form of Euler’s equations and its subsequent approximations.

2.1 Theory

The theory behind a general quaternionic structure for a given three-dimensional vorticity equation is discussed below. The corresponding mathematical formulation of this theory is then derived.

If there exists a three-dimensional vorticity representation of a set of equations that define a particular fluid system then under certain conditions a basic or inherent quaternionic structure exists in the evolution of transformed variables from the
initial vorticity equation. Each individual approximation either retains or causes
the breakdown of this quaternionic structure; and specific examples of these cases
will be given in Chapters 3 and 4. To get a complete picture, however, of the
quaternionic structure in these transformed variables an explicit expression for the
evolution of the vorticity stretching vector is required and must be derived for each
system of equations under consideration. Finally, due to this transformation of vari-
ables, there are now four and not three equations and therefore a constraint equation
is needed. This equation provides a relationship between the dependent variables
in the flow.

2.2 Mathematical derivation

2.2.1 Evolution equations for the stretching rate and the align-
ment vector

The mathematical reasoning of the theory given in the previous section begins as
follows:

Consider a general vorticity equation of the form

$$\frac{D\omega}{Dt} = \sigma, \quad (2.1)$$

where $\omega$ is the vorticity (which is not necessarily $\nabla \times u$ but is taken to be any
vector $\omega$) of the flow with velocity $u$ and $\sigma$ is the vorticity stretching vector, where
each can be thought of as a function of position and time. The material derivative
is defined by

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u \cdot \nabla, \quad (2.2)$$

where $u$ is the velocity field and $\nabla$ is the three-dimensional gradient operator. Al-
though the phrase “vorticity stretching vector” is used to signify the right hand
side of (2.1) it should be noted that for a more specific vorticity equation the right
hand side may also include tilting and baroclinic terms. The expression “vorticity
stretched vector” is used here for convenience and for the simple fact that in the first couple of examples in Chapter 3 the right hand side is exactly given by the vorticity stretching vector alone. However, $\sigma$ will always be defined as the right hand side of the vorticity equation.

Recall from the definition of the scalar product that the relationship between the vorticity $\omega$ and the corresponding unit vector $|\omega|$, is given by,

$$\omega \cdot \omega = |\omega|^2. \quad (2.3)$$

Taking the material derivative of (2.3) gives

$$\omega \cdot \frac{D\omega}{Dt} = |\omega|\frac{D}{Dt}|\omega|. \quad (2.4)$$

Substituting the general form of the vorticity equation seen in (2.1) gives

$$|\omega|\frac{D}{Dt}|\omega| = \omega \cdot \sigma. \quad (2.5)$$

Dividing through (2.5) by $|\omega|$ gives

$$\frac{D}{Dt}|\omega| = \alpha|\omega|; \quad (2.6)$$

this is the evolution equation for the vorticity magnitude. The variable $\alpha$ in (2.6) is given explicitly by

$$\alpha(x, t) = \frac{\omega \cdot \sigma}{\omega \cdot \omega}. \quad (2.7)$$

This variable $\alpha$ is known as the stretching rate for the flow and is simply the material derivative of the logarithm of the vorticity magnitude. For $\alpha > 0$ there is vortex stretching and for $\alpha < 0$ there is vortex compression within the flow. To fully take into account how the vorticity orientates itself with the vorticity stretching vector, the following alignment vector $\chi$, or as it is referred to in some literature (Galanti et al. (1997), Gibbon et al. (2000) and Gibbon (2002)) - the mis-alignment vector, is defined as
Chapter 2. Quaternionic structure of a general 3D vorticity equation

\[ \chi(x, t) = \frac{\omega \times \sigma}{\omega \cdot \omega}. \]  

(2.8)

A link between these two variables is seen by considering the local angle \( \phi(x, t) \) that lies between the vorticity vector \( \omega \) and the vorticity stretching vector \( \sigma \) at the point \( x \) at time \( t \). The angle \( \phi \) is defined by

\[ \tan \phi(x, t) = \frac{\left| \omega \times \sigma \right|}{\omega \cdot \sigma} = \frac{\chi}{\alpha}, \]  

(2.9)

where \( |\chi| \) is the alignment vector magnitude (\( |\chi|^2 = \chi \cdot \chi \)). The \( \chi \)-vector arises naturally in the dynamics of the vorticity and takes a similar form to that of the stretching rate given in equation (2.6). Instead of transforming the vorticity equation, as we did when deriving an expression for \( \alpha \), the evolution of the following vorticity unit vector \( \hat{\omega} = \frac{\omega}{|\omega|} \) is considered, and takes the form

\[ \frac{D\hat{\omega}}{Dt} = \frac{|\omega|}{\frac{D}{Dt} |\omega|} \frac{D}{Dt} \omega - \omega \frac{D}{Dt} |\omega| = \frac{(\sigma - \alpha \omega) |\omega|}{\omega \cdot \omega}, \]  

(2.10)

using the definitions of \( \alpha \), given in equation (2.7), and \( \hat{\omega} \) to give

\[ \frac{D\hat{\omega}}{Dt} = \frac{\sigma |\omega| - (\omega \cdot \sigma) \hat{\omega}}{\omega \cdot \omega}. \]  

(2.11)

The numerator on the right-hand side of (2.11) can be re-written as

\[ \hat{\omega} \times (\omega \times \sigma) - (\hat{\omega} \times \omega) \times \sigma = \hat{\omega} (\omega \cdot \sigma) - \sigma (\omega \cdot \hat{\omega}), \]  

(2.12)

the second term in equation (2.12) is zero so \( (\omega \cdot \sigma) \hat{\omega} = \hat{\omega} \times (\omega \times \sigma) + \sigma (\omega \cdot \hat{\omega}) \), substituting this result back into equation (2.12) gives

\[ \frac{D\hat{\omega}}{Dt} = -\hat{\omega} \times \left( \frac{\omega \times \sigma}{\omega \cdot \omega} \right). \]  

(2.13)

Using the definition of the alignment vector given in equation (2.8) gives, first, an alternate form for the vortex stretching in terms of the \( (\alpha, \chi) \) variables

\[ \sigma = \alpha \omega + \chi \times \omega, \]  

(2.14)
and, second, it gives the following expression for the evolution of the vorticity unit vector $\hat{\omega}$

$$\frac{D\hat{\omega}}{Dt} = \chi \times \hat{\omega}. \quad (2.15)$$

These two results, equations (2.6) and equation (2.15), show that $\alpha$ and $\chi$ are the rates of change of the vorticity magnitude and direction respectively.

Consider the case of the right-hand side of equation (2.15) being zero. This implies that the quantities $\hat{\omega}$ and $\chi$ are parallel or one is zero at some time within the flow. Therefore the local angle, $\hat{\phi}$, between these quantities is either 0 or $\pi$, and this would imply that the vorticity unit vector $\hat{\omega}$ is a conserved quantity. However, from basic vector analysis and Figure 2.1, which shows the relative positions of key quantities that appear in the flow, $\chi \cdot \hat{\omega} = 0$, and so the quantities, if non-zero, are orthogonal, but of different magnitude, and the only case when they would be parallel is the trivial case of either quantity being zero. The right-hand side of
equation (2.15) can be re-written as

$$\chi \times \hat{\omega} = |\chi| |\hat{\omega}| \sin \hat{\phi} \cdot \hat{n},$$

where \(\hat{n}\) is a unit vector normal to \(\hat{\omega}\) and \(\chi\). From the earlier analysis, it was found that these two quantities are in fact orthogonal and so \(\hat{\phi} = \pm \pi/2\), and from the definition of \(\hat{\omega}\), its magnitude is equal to 1. Equation (2.15) can be simplified to

$$\frac{D\hat{\omega}}{Dt} = |\chi| \hat{n}. \quad (2.16)$$

The evolution of the unit vorticity vector \(\hat{\omega}\) is equal to the magnitude of the alignment vector in the direction orthogonal to the quantities \(\chi\) and \(\hat{\omega}\). Figure 2.1 further shows that the vectors \(\{\omega, \chi, \omega \times \chi\}\) form an ortho-normal co-ordinate system and from this it is clear that the vorticity stretching term \(\sigma\) can be decomposed from the two orthogonal vectors \(\omega\) and \(\chi \times \omega\) and hence the result seen in equation (2.14). A more detailed consideration of the role that the stretching rate \(\alpha\) and the alignment vector \(\chi\) play in the study of vorticity was discussed in the Introduction and will be further re-iterated when specific flow regimes are considered in later chapters.

Taking the material derivative of equation (2.7) gives

$$\frac{D\alpha}{Dt} = \frac{D\omega}{\omega \cdot \omega} \cdot \sigma + \frac{\omega \cdot D\sigma}{\omega \cdot \omega} - (\omega \cdot \omega)^{-2} \left\{ \frac{D\omega}{Dt} \cdot \omega + \omega \cdot \frac{D\omega}{Dt} \right\} (\omega \cdot \sigma),$$

$$= \frac{\sigma \cdot \sigma}{\omega \cdot \omega} - 2 \frac{(\omega \cdot \sigma)^2}{(\omega \cdot \omega)^2} + \frac{\omega \cdot D\sigma}{\omega \cdot \omega}. \quad (2.17)$$

From the definition of the scalar and vector product, \(\omega \cdot \sigma = |\omega| |\sigma| \cos \phi\) and \(\omega \times \sigma = |\omega| |\sigma| \sin \phi \cdot \hat{n}\), re-arranging these expressions gives

$$\cos^2 \phi = \frac{(\omega \cdot \sigma)^2}{|\omega|^2 |\sigma|^2} \quad \text{and} \quad \sin^2 \phi = \frac{|\omega \times \sigma|^2}{|\omega|^2 |\sigma|^2},$$

the numerator of the first term in equation (2.17) can be re-written as

$$|\sigma|^2 = \frac{(\omega \cdot \sigma)^2}{\omega \cdot \omega} + \frac{|\omega \times \sigma|^2}{\omega \cdot \omega}.$$
and so the first term in equation (2.17) is

\[
\frac{\sigma \cdot \sigma}{\omega \cdot \omega} = \left( \frac{\omega \cdot \sigma}{\omega \cdot \omega} \right)^2 + \left( \frac{\omega \times \sigma}{\omega \cdot \omega} \right)^2
= \alpha^2 + |\chi|^2.
\]

This result can also be obtained by consider a corresponding expression for $|\sigma|^2$ in equation (2.14). Combining this result with equations (2.7) and (2.8) gives the following expression for the evolution of the stretching rate

\[
\frac{D\alpha}{Dt} = |\chi|^2 - \alpha^2 + |\omega|^{-2} \left( \omega \cdot \frac{D\sigma}{Dt} \right).
\]

(2.18)

Similarly for the alignment vector $\chi$

\[
\frac{D\chi}{Dt} = \frac{D\omega}{Dt} \times \sigma + \omega \times \frac{D\sigma}{Dt} \omega \cdot \omega - (\omega \cdot \omega)^{-2} \left\{ \frac{D\omega}{Dt} \cdot \omega + \omega \cdot \frac{D\omega}{Dt} \right\} (\omega \times \sigma).
\]

(2.19)

The first term in (2.19) is identically zero and the remaining two terms give the following expression for the evolution of the alignment vector

\[
\frac{D\chi}{Dt} = -2\chi\alpha + |\omega|^{-2} \left( \omega \times \frac{D\sigma}{Dt} \right).
\]

(2.20)

From equations (2.18) and (2.20) it is clear that to get an explicit expression for the evolution of the stretching rate $\alpha$ and the alignment vector $\chi$ a corresponding expression for the evolution of the vorticity stretching vector $\sigma$ has to be derived. This, however, can only be done when a particular flow regime governed by a particular set of equations is defined. Even with these general expressions for $\alpha$ and $\chi$ the right hand sides of equations (2.18) and (2.20) suggest a structure in $\mathbb{R}^4$ based on quaternions. At this juncture it would be wise to give a detailed explanation of quaternions and their corresponding algebra. It will then be possible to proceed and explain exactly the meaning of the statement that a set of equations, for example, the evolution equations for the stretching rate and the alignment vector or even the Euler equations, have a “quaternionic structure”.
2.2.2 Quaternions and their corresponding algebra

A quaternion is simply a generalisation of complex numbers to \( \mathbb{R}^4 \). Recall that a complex number takes the form \( z = x + iy \) where \( x \) and \( y \) are real numbers and \( i^2 = -1 \). A quaternion is a generalised complex number, \( q \), consisting of four components such that

\[
q = q_0 + iq_1 + jq_2 + kq_3,
\]

where the \( q_i \) are real numbers and \( i, j, k \) are an extension of complex numbers known as hypercomplex numbers, with corresponding multiplication rules

\[
i^2 = j^2 = k^2 = -1, \\
i j = k = -ji, \ldots \text{cyclical permutations.}
\]  

(2.22)

Quaternions form a group \( Q \) of order \(|Q| = 8\) consisting of elements \( \{\pm I, \pm e_1, \pm e_2, \pm e_3\} \) where \( I \) is the \( 2 \times 2 \) identity matrix and each basis \( e_i \) is given by

\[
\pm e_i = \mp i\sigma_i,
\]

(2.23)

where \( \sigma_i \) are the Pauli matrices

\[
\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},
\]

(2.24)

and the results \( e_1^2 = e_2^2 = e_3^2 = -I \) and \( e_1 e_2 = e_3 = -e_2e_1, \ldots \), isomorphic to (2.22) hold. For any two quaternions \( a = a_0 + ia_1 + ja_2 + ka_3 \) and \( b = b_0 + ib_1 + jb_2 + kb_3 \), the additive rule for quaternions is quite simple and is achieved by adding each “real” part and each different “hypercomplex” part together to give \( a + b = (a_0 + b_0) + i(a_1 + b_1) + j(a_2 + b_2) + k(a_3 + b_3) \). Multiplication is a little more advanced but by following the rules set out in (2.22) and considering each element of \( a \) acting on each element of \( b \) in turn their product is given by
\[ ab = (a_0 + ia_1 + ja_2 + ka_3) (b_0 + ib_1 + jb_2 + kb_3), \]
\[ = a_0b_0 - (a_1b_1 + a_2b_2 + a_3b_3) + i (a_0b_1 + a_1b_0 + a_2b_3 - a_3b_2) \quad (2.25) \]
\[ + j (a_0b_2 + a_2b_0 + a_3b_1 - a_1b_3) + k (a_0b_3 + a_3b_0 + a_1b_2 - a_2b_1). \]

The majority of the mathematics in this research takes place on \( \mathbb{R}^4 \) and so an algebra on \( \mathbb{R}^4 \) is needed. For convenience we are going to use the representation of column vectors to define elements in this space. Consider the following unit 4-vectors \( 1 = (1, 0, 0, 0)^T, i = (0, 1, 0, 0)^T, j = (0, 0, 1, 0)^T, k = (0, 0, 0, 1)^T \), this representation can be thought of in terms of a map from the unit vectors \( (1, i, j, k) \) to the corresponding elements of the group \( Q \) i.e. \( (I, e_1, e_2, e_3) \). Define corresponding multiplication laws as

\[
i \otimes i = j \otimes j = k \otimes k = -1, \\
i \otimes j = k = -j \otimes i, \ldots \text{cyclical permutations.} \quad (2.26)\]

This algebra is isomorphic to the quaternionic algebra seen in equation (2.22). Define general 4-vectors \( q_i \) by

\[ q_i = \begin{pmatrix} \alpha_i \\ \chi_i \end{pmatrix}. \]

Then the product of any two 4-vectors, \( q_1 \) and \( q_2 \), using the definitions, in (2.26), and noting the result (2.25), is given by

\[
q_1 \otimes q_2 = \begin{pmatrix} \alpha_1\alpha_2 - \chi_1 \cdot \chi_2 \\ \alpha_1\chi_2 + \alpha_2\chi_1 + \chi_1 \times \chi_2 \end{pmatrix}. \quad (2.27)
\]

It is now possible to apply these results regarding the algebraic properties of quaternions to our two equations for the evolution of the stretching rate \( \alpha \) and the alignment vector \( \chi \).
Chapter 2. Quaternionic structure of a general 3D vorticity equation

2.2.3 An inherent/basic quaternionic structure

Using the multiplication rule (2.27) for two arbitrary 4-vectors and defining three additional 4-vectors $q, s_1$ and $s_2$ as

$$ q = \begin{pmatrix} \alpha \\ \chi \end{pmatrix}, \quad s_1 = \begin{pmatrix} 0 \\ \frac{D\sigma}{Dt} \end{pmatrix}, \quad s_2 = \begin{pmatrix} 0 \\ |\omega|^{-2}\omega \end{pmatrix}, \quad (2.28) $$

it is possible to re-write the two equations (2.18) and (2.20) as a single evolution equation with respect to the new variable $q$ as

$$ \frac{Dq}{Dt} + q \otimes q + s_1 \otimes s_2 = 0. \quad (2.29) $$

This is a quaternionic equation for the evolution of the 4-vector $q$.

Let us summarise what has been done so far: given a general set of momentum equations that define a particular flow regime it is possible to transform these equations to a corresponding set of vorticity equations. Through consideration of the equations of motion for the vorticity magnitude and direction a further set of variables, which provide information regarding the stretching and alignment of the vorticity, can be derived. A new 4-vector $q$, a combination of the $(\alpha, \chi)$ variables, can then be explicitly written as the following quaternion

$$ q = 1\alpha + i\chi_1 + j\chi_2 + k\chi_3, \quad (2.30) $$

where $\chi_i$ are the three components of the vector $\chi$. Similar expressions can also be used to express the vectors $s_1$ and $s_2$ as quaternions. It is interesting to note that $s_1$ and $s_2$ are simply 3-vectors written in 4-vector form with no scalar term and so in the language of complex numbers would be defined as being purely imaginary (i.e. the only non-zero coefficients are in the $i, j, k$ terms). Expanding the direct product terms $q \otimes q$ and $s_1 \otimes s_2$ using the multiplication rules of quaternionic algebra given in equation (2.26) and decomposing these two products into their real (or scalar $\alpha$) and imaginary (or vector $\chi$) parts, the two equations (2.18) and (2.20) are obtained. This is exactly what is meant when a set of momentum equations are said to have
Chapter 2. Quaternionic structure of a general 3D vorticity equation

a quaternionic structure, namely, that the form of the evolution equations for the $\alpha$ and $\chi$ variables have an algebraic structure in $\mathbb{R}^4$, namely that of quaternions and their associated algebra.

Considering equation (2.29) in greater detail then the product term $\mathbf{q} \otimes \mathbf{q}$ can be thought of as an equation independent term that will be present in all subsequent approximations provided that the equations for the evolution of $(\alpha, \chi)$ exist and are non-zero. Although the physical and mathematical components of the vorticity, and hence the vorticity stretching, will change with each successive approximation the definitions of $\alpha$ and $\chi$ given in equations (2.7) and (2.8) respectively will be unaltered. This is due to the fact that the derivations of the stretching rate and alignment vector hold for each system of equations, as $\alpha$ and $\chi$ are approximately, neglecting the scalar factor of $|\omega|^{-2}$, the scalar and vector products respectively of $\omega$ and $\sigma$ regardless of the actual physical form of the components of these two quantities.

The second product term in equation (2.29), $\mathbf{s}_1 \otimes \mathbf{s}_2$, is the part of the evolution equation for $\mathbf{q}$ that will change with each successive approximation and can therefore be thought of as the equation dependent term in (2.29). This is because an explicit expression for $\sigma$, and its derivative, is required to calculate the product term $(\mathbf{s}_1 \otimes \mathbf{s}_2)$ for each set of equations. As each approximation is considered the form of the vorticity stretching vector changes and hence so does the equation dependent term. Once this product term is known an explicit expression for the evolution of the 4-vector $\mathbf{q}$ in terms of key properties (i.e. pressure, vorticity, temperature etc.) in the flow is known.

One key point to make is that when the full form of the $\frac{D\mathbf{q}}{Dt}$ equation is derived that does not mean that the problem is closed, in fact, a constraint equation is needed and the justification for this and further details are mentioned in section 2.5.
2.3 Alternative derivation of the quaternionic structure

The results derived for the evolution of the stretching rate $\alpha$, the vortex alignment vector $\chi$, and the subsequent quaternionic formulation given in the previous section can be derived by defining a further vorticity 4-vector $\mathbf{w}$, which is very similar to the vector $\mathbf{s}_2$

$$\mathbf{w} = \begin{pmatrix} 0 \\ \mathbf{\omega} \end{pmatrix}. \quad (2.31)$$

**Theorem** The vorticity 4-vector $\mathbf{w}$ and the 4-vector $\mathbf{q}$ satisfy the following quaternionic relationship

$$\frac{D\mathbf{w}}{Dt} = \mathbf{q} \otimes \mathbf{w}. \quad (2.32)$$

**Proof** The right-hand side of equation (2.32) is explicitly given by

$$\mathbf{q} \otimes \mathbf{w} = \begin{pmatrix} \alpha \\ \chi \end{pmatrix} \otimes \begin{pmatrix} 0 \\ \mathbf{\omega} \end{pmatrix} = \begin{pmatrix} -\chi \cdot \mathbf{\omega} \\ \alpha \mathbf{\omega} + \chi \times \mathbf{\omega} \end{pmatrix}. \quad (2.33)$$

The scalar component is zero, and once again using the result given in equation (2.14) we find

$$\begin{pmatrix} 0 \\ \alpha \mathbf{\omega} + \chi \times \mathbf{\omega} \end{pmatrix} = \begin{pmatrix} 0 \\ \sigma \end{pmatrix} = \frac{D\mathbf{w}}{Dt}. \quad (2.34)$$

This is equivalent to the initial vorticity equation given in equation (2.1).

First take the derivative of (2.32) to give

$$\frac{D^2\mathbf{w}}{Dt^2} = \frac{D\mathbf{q}}{Dt} \otimes \mathbf{w} + \mathbf{q} \otimes \frac{D\mathbf{w}}{Dt} = \frac{D\mathbf{q}}{Dt} \otimes \mathbf{w} + \mathbf{q} \otimes (\mathbf{q} \otimes \mathbf{w}), \quad (2.33)$$

then take the quaternionic product of equation (2.33) with $\mathbf{w}$ to give

$$\frac{D^2\mathbf{w}}{Dt^2} \otimes \mathbf{w} = \left( \frac{D\mathbf{q}}{Dt} \otimes \mathbf{w} \right) \otimes \mathbf{w} + \{ \mathbf{q} \otimes (\mathbf{q} \otimes \mathbf{w}) \} \otimes \mathbf{w}. \quad (2.34)$$
Chapter 2. Quaternionic structure of a general 3D vorticity equation

This can be simplified by using the result that \( \mathbf{w} \otimes \mathbf{w} = (0, \omega)^T \otimes (0, \omega)^T = -|\omega|^2 (1, 0)^T \), so that

\[
\frac{D\mathbf{q}}{Dt} + \mathbf{q} \otimes \mathbf{q} + \frac{1}{\mathbf{w} \cdot \mathbf{w}} \frac{D^2 \mathbf{w}}{Dt^2} \otimes \mathbf{w} = 0. \tag{2.35}
\]

This equation is equivalent to (2.29) with the equation dependent term taking the form \( |\omega|^{-2} (D^2 \mathbf{w} / Dt^2) \otimes \mathbf{w} \) but this form does benefit from the ease of calculation seen above. However, as in (2.29) the equation is not closed until the explicit form for the evolution of the vorticity evolution vector is calculated.

### 2.4 Corresponding complex structure

Recall the earlier discussion on quaternions where it was mentioned that there exists an obvious relationship between complex numbers and quaternions; it was stated that quaternions are a generalisation of complex numbers to \( \mathbb{R}^4 \). Is it therefore possible to re-write the evolution equations (2.18) and (2.20) in terms of complex variables? It is and can be achieved by reducing the evolution equation for the 3-vector \( \chi \) to one singular equation in terms of the scalar \( |\chi| \). Taking the scalar product of \( \chi \) with equation (2.20) gives

\[
\chi \cdot \frac{D\chi}{Dt} = |\chi| \frac{D|\chi|}{Dt} = -2|\chi|^2 \alpha + \chi \cdot |\omega|^{-2} \left( \omega \times \frac{D\sigma}{Dt} \right),
\]

re-arranging this gives the following expression for the evolution of the scalar \( |\chi| \) together with the equation for \( \alpha \)

\[
\frac{D\alpha}{Dt} = |\chi|^2 - \alpha^2 + |\omega|^{-2} \left( \omega \cdot \frac{D\sigma}{Dt} \right),
\]

\[
\frac{D|\chi|}{Dt} = -2|\chi| \alpha + |\omega|^{-2} \hat{\chi} \cdot \left( \omega \times \frac{D\sigma}{Dt} \right), \tag{2.36}
\]

where \( \hat{\chi} \) is the unit alignment vector.

Introducing the following complex variables \( \zeta_c \) and \( \psi_c \)
Chapter 2. Quaternionic structure of a general 3D vorticity equation

\[ \zeta_c = \alpha + i|\chi|, \quad \psi_c = |\omega|^{-2} (\omega \cdot \frac{D\sigma}{Dt}) + i|\omega|^{-2} (\hat{\chi} \cdot \omega \times \frac{D\sigma}{Dt}), \quad (2.37) \]

it is possible to re-write equations (2.36) using the variables defined in (2.37) to obtain the following complex Riccati equation

\[ \frac{D\zeta_c}{Dt} + \zeta_c^2 = \psi_c. \quad (2.38) \]

This Riccati equation can be linearised by introducing a further complex variable \( \gamma_c \) given by the substitution

\[ \zeta_c = \frac{1}{\gamma_c} \frac{D\gamma_c}{Dt}. \quad (2.39) \]

This transforms equation (2.38) into

\[ \frac{D}{Dt} \left( \frac{1}{\gamma_c} \frac{D\gamma_c}{Dt} \right) + \left( \frac{1}{\gamma_c} \frac{D\gamma_c}{Dt} \right)^2 = \psi_c, \]

\[ \frac{1}{\gamma_c} \frac{D^2\gamma_c}{Dt^2} - \frac{1}{\gamma_c^2} \frac{D\gamma_c}{Dt} \frac{D\gamma_c}{Dt} + \left( \frac{1}{\gamma_c} \frac{D\gamma_c}{Dt} \right)^2 = \psi_c, \]

hence giving the following zero eigenvalue (scalar) Schrödinger equation with potential \( \psi_c \)

\[ \frac{D^2\gamma_c}{Dt^2} = \psi_c \gamma_c. \quad (2.40) \]

This equation and its solutions will be discussed in greater detail in the next chapter when the focus shifts from this general case to flow governed by the three-dimensional incompressible Euler equations. As with the corresponding quaternionic formulation, this equation is not “closed” until the exact form of the potential \( \psi_c \) is explicitly calculated.

### 2.5 Mathematical constraints

From the mathematical derivations of the previous sections it is vital to note that the fundamental transformation is from a vorticity equation with three scalar equations
Chapter 2. Quaternionic structure of a general 3D vorticity equation

(for the three components of the vorticity) to a single equation for the stretching rate $\alpha$ and three further scalar equations for the alignment vector $\chi$. Hence a further constraint equation is required. This means that there exists one additional equation of motion and so a constraint equation will provide not only a relationship between certain dependent variables attributed to the flow but also information regarding the relationship between the equation dependent and independent terms seen in (2.29). This constraint, however, needs to be derived for each set of equations under consideration and its exact form will become clear in the next chapter.

2.6 Conditions and limitations of the theory

Before considering particular flow regimes a mention must be made of the limitations governing the theory outlined above. Firstly, it is required that the particular flow under discussion can be expressed in a three-dimensional vorticity framework and so, in general, the theory can not be applied to two-dimensional systems e.g. the shallow water primitive equations, although there may exist a corresponding complex structure in this form. Secondly, the theory can not be applied if the vorticity is conserved and hence the right-hand side of the vorticity equation is zero, i.e. $\sigma = 0$. One immediate example of this that comes to mind, although dealt with in the first condition, is the case of incompressible, two-dimensional Euler. Finally, the quaternionic structure would be trivial if either of the key quantities ($\alpha, \chi$) were zero. The stretching rate being zero would imply that the vorticity and vorticity stretching are orthogonal and there would be rotation and no stretching. The alignment vector being zero would imply perfect alignment (or anti-alignment) between the vorticity and vorticity stretching and only vortex stretching would occur. The theory can, however, be applied to systems where there is no prognostic equation for all three velocity components e.g. the hydrostatic, primitive equations as it is still possible to construct a three dimensional vorticity equation. This last part must be stressed as the main condition for the retention of a quaternionic structure with respect to a set of momentum equations, that there exists a three-dimensional vor-
ticity equation with corresponding non-zero vorticity stretching vector.

2.7 Summary

In this chapter the \((\alpha, \chi)\) variables have been introduced in terms of a general vorticity \(\omega\) and vorticity stretching vector \(\sigma\). By considering the Lagrangian derivatives of the vorticity magnitude \(|\omega|\) and vorticity unit vector \(\hat{\omega}\) we find

\[
\frac{D|\omega|}{Dt} = \alpha|\omega|, \quad \frac{D\hat{\omega}}{Dt} = \chi \times \hat{\omega}; \tag{2.41}
\]

these results indicate that the \((\alpha, \chi)\) variables defined in (2.7) are the rate of change of vorticity magnitude and direction respectively. The evolution of these variables can be represented by the three 4-vectors \(q, s_1, s_2\) in terms of the following equation

\[
\frac{Dq}{Dt} + q \otimes q + s_1 \otimes s_2 = 0. \tag{2.42}
\]

A detailed discussion of equation (2.42) took place that not only highlighted the context of this equation in the general framework of quaternion algebra but also the specific labelling of the two terms as the equation dependent and independent terms. A further derivation of this quaternionic equation for \(q\) was seen by taking the derivative of the 4-vector representation, \(w = (0, \omega)^T\), of the general vorticity \(\omega\), which satisfies

\[
\frac{Dw}{Dt} = q \otimes w, \tag{2.43}
\]

and by considering the derivative of this expression gives

\[
\frac{Dq}{Dt} + q \otimes q + \frac{1}{w \cdot w} \frac{D^2w}{Dt^2} \otimes w = 0. \tag{2.44}
\]

which is equivalent to (2.42). The corresponding complex structure was also derived and discussed.

Finally, the need to close the system by obtaining a suitable constraint equation and the conditions and limitations of such a general theory were highlighted.
The next chapter considers, partly in review, but also with respect to the results seen in this chapter, the mathematics of the quaternionic structure, and corresponding results, for the particular case of the three-dimensional incompressible Euler equations.
Chapter 3

The inertial incompressible Euler equations

The aim in this chapter is to begin applying the theory set out in Chapter 2 to particular flow regimes. Although the general form of the equation of motion for the 4-vector $\mathbf{q}$ is given in equation (2.29), for each set of equations the following calculations are required:

- derive the vorticity equation from the corresponding set of momentum equations;
- find an explicit expression for the evolution of the vorticity stretching vector i.e. $\frac{\partial \mathbf{\sigma}}{\partial t}$;
- calculate the corresponding constraint equation.

The complications associated with trying to calculate such expressions with respect to particular sets of momentum equations will be discussed in detail and similarly the solutions to such problems will be given where known. Furthermore, by dealing with specific flow regimes, the role of the $\alpha$ and $\chi$ variables in understanding such flows, and possible practical applications of these variables, will also be discussed. In this chapter we begin with the inertial incompressible Euler equations.
Chapter 3. The inertial incompressible Euler equations

3.1 Equations of motion

The three-dimensional momentum equation that relates the velocity $u(x,t)$ to the pressure $p(x,t)$ and the density $\rho(x,t)$ for an inviscid fluid in an inertial reference frame is given by

$$\frac{Du}{Dt} = -\frac{1}{\rho} \nabla p,$$

where the material derivative is defined in (2.2). Regardless of the fluid under consideration each fluid element must conserve its mass as it moves throughout the flow. This corresponds mathematically to

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho u) = 0.$$ (3.2)

For a flow of constant density (3.2) simplifies to

$$\nabla \cdot u = 0.$$ (3.3)

This constraint, known as the incompressibility condition, simply says that no fluid particle can change its volume as it moves. Equations (3.1) and (3.3) are known as the three-dimensional Euler equations for an incompressible fluid in an inertial reference frame. The momentum equation (3.1) can be simplified further for a constant density fluid. The density can be incorporated into the pressure gradient term as $-\nabla (p/\rho)$, and either by re-defining $(p/\rho)$ or simply, without any loss of generality, setting $\rho = 1$, and expanding the material derivative, equation (3.1) becomes

$$\frac{\partial u}{\partial t} + u \cdot \nabla u = -\nabla p.$$ (3.4)

The nonlinear advection term in equation (3.4) can be expressed as $u \cdot \nabla u = (\nabla \times u) \times u + \frac{1}{2} \nabla (|u|^2)$ giving the following alternative form for equation (3.4)

$$\frac{\partial u}{\partial t} + \omega \times u = -\nabla \left( p + \frac{1}{2} |u|^2 \right),$$ (3.5)
Chapter 3. The inertial incompressible Euler equations

where \( \omega \) is the vorticity vector \( \omega = \nabla \times u \). Taking the curl of equation (3.5) together with standard vector identities (see glossary) gives

\[
\frac{\partial \omega}{\partial t} + (u \cdot \nabla) \omega - (\omega \cdot \nabla) u + \omega (\nabla \cdot u) - u (\nabla \cdot \omega) = 0. \tag{3.6}
\]

The term \( \omega (\nabla \cdot u) \) is zero due to the incompressibility constraint and \( u (\nabla \cdot \omega) \) is zero from the fact that for any continuous, twice-differentiable three-dimensional vector field \( A \), \( \text{div} \text{curl} A = 0 \) respectively. Therefore, equation (3.6) simplifies to

\[
\frac{\partial \omega}{\partial t} + u \cdot \nabla \omega - \omega \cdot \nabla u = 0, \tag{3.7}
\]

or in terms of the material derivative the evolution of the vorticity is given by

\[
\frac{D\omega}{Dt} = (\omega \cdot \nabla) u. \tag{3.8}
\]

For the incompressible Euler equations the vorticity stretching vector is given explicitly by \( \sigma = (\omega \cdot \nabla) u \) which can be re-written as \( \sigma = (\omega \cdot \nabla) u = S\omega \) where \( S \), a function of the velocity field \( u \), is the strain matrix. The \( ij \)-th element of the strain matrix is given by \( S = \frac{1}{2} (u_{i,j} + u_{j,i}) \), where \( u_{i,j} \) denotes the partial derivative of the \( i \)-th component of the velocity field \( u \) with respect to the \( j \)-th component of the co-ordinate variable \( x \). In matrix form this is given by

\[
S = \frac{1}{2} \begin{pmatrix}
\frac{\partial u}{\partial x} & \frac{1}{2} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) & \frac{1}{2} \left( \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right) \\
\frac{1}{2} \left( \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) & \frac{\partial v}{\partial y} & \frac{1}{2} \left( \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right) \\
\frac{1}{2} \left( \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \right) & \frac{1}{2} \left( \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} \right) & \frac{\partial w}{\partial z}
\end{pmatrix}. \tag{3.9}
\]

The strain matrix \( S \) constitutes the symmetric part of the velocity gradient matrix \( u_{i,j} \). The reason the vorticity stretching vector takes the form \( S\omega \) is because the vorticity, which is simply the curl of the velocity field, sees only the symmetric part of the matrix \( u_{i,j} \).
### 3.2 The stretching rate and vorticity alignment

Now that the form of the vorticity equation corresponding to the case of the threedimensional incompressible Euler equations has been derived, the stretching rate is given explicitly by

\[
\alpha = \frac{\omega \cdot \sigma}{\omega \cdot \omega} = \frac{\omega \cdot (\omega \cdot \nabla) u}{\omega \cdot \omega} = \frac{\omega \cdot S \omega}{\omega \cdot \omega},
\]

similarly for the alignment vector / spin rate \( \chi \)

\[
\chi = \frac{\omega \times \sigma}{\omega \cdot \omega} = \frac{\omega \times (\omega \cdot \nabla) u}{\omega \cdot \omega} = \frac{\omega \times S \omega}{\omega \cdot \omega}.
\]

In this form the stretching rate and the vorticity alignment vector provide a relationship between the vorticity and the strain matrix \( S \). Recall that if the stretching rate is positive then there is vortex stretching and negative values lead to vortex compression. Furthermore, Constantin (1994) has shown that it is possible to obtain an integral expression for the stretching rate in terms of the vorticity. This is achieved by considering the velocity in terms of the vorticity by the Biot-Savart law

\[
u(x) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{x-y}{|x-y|^3} \times \omega(y) \, dy,
\]

by differentiating equation (3.12) an expression for the full gradient of the velocity is obtained. Splitting this expression into its symmetric and anti-symmetric parts gives an explicit integral representation for the strain matrix \( S(x) \). Finally from this expression, the corresponding stretching rate can be derived. This formula for \( \alpha(x) \) is

\[
\alpha(x) = \frac{3}{4\pi} \text{P.V.} \int_{\mathbb{R}^3} D(\hat{y}, \hat{\omega}(x+y), \hat{\omega}(x)) |\omega(x+y)| \frac{dy}{|y|^3},
\]

where \( \hat{y} = y/|y| \) and \( D \) is

\[
D(e_1, e_2, e_3) = (e_1 \cdot e_3) (\text{Det}(e_1, e_2, e_3)).
\]
The determinant in (3.14) is equivalent to that of a matrix whose columns are the three unit vectors $e_1, e_2, e_3$. The Biot-Savart type integral (3.13) relates the stretching rate to a prism of vectors with edges equal to $\hat{y}, \hat{\omega}(x + y), \hat{\omega}(x)$ that characterise the (relative) alignment of neighbouring vortex lines.

In Gibbon (2002) the relation between these variables and the spectrum of $S$ is discussed. If $\lambda_i$ are the eigenvalues of $S$ then the incompressible constraint (3.3) implies that $\text{Tr} S = \lambda_1 + \lambda_2 + \lambda_3 = 0$. If it assumed that $\lambda_1 \leq \lambda_2 \leq \lambda_3$ then $\lambda_3 \geq 0$, $\lambda_1 \leq 0$ and $\lambda_2$ is of variable sign. From equation (3.10) $\alpha$ is a (Rayleigh quotient) estimate for an eigenvalue of $S$ and is bounded by $\lambda_1 \leq \alpha \leq \lambda_3$.

3.2.1 The role of the local angle $\phi$

In the previous chapter, the local angle, between the vorticity and the vorticity stretching $\phi$ was introduced. For the incompressible Euler equations this is explicitly the angle between the vorticity $\omega$ and $S\omega$. For a range of angles it is possible to get a greater understanding of the behaviour of $\alpha$ and $\chi$ and hence the nature of the vorticity and $S\omega$.

1. $\phi = 0$ implies $|\chi| = 0$ therefore there is perfect alignment between $\omega$ and $S\omega$ and hence $\alpha > 0$, only vorticity stretching occurs.

2. $\phi = \pi/2$ implies $\alpha = 0$ hence the vorticity magnitude is conserved and $\omega$ and $S\omega$ are orthogonal. The vorticity can only rotate but not stretch.

3. $\phi = \pi$ once again implies $|\chi| = 0$ although for this case there is anti-alignment between $\omega$ and $S\omega$ and hence $\alpha < 0$. The vorticity will in fact collapse rapidly at this limit.

4. $0 < \phi < \pi/2$ implies $\tan \phi > 0$ and hence the stretching rate is positive. In this case there will be both vortex stretching and rotation.

5. $\pi/2 < \phi < \pi$ implies $\tan \phi < 0$ and hence the stretching rate is negative. Once again there will be rotation but this be accompanied with vortex compression which will become rapid as $\phi \to \pi$. 
3.3 Equivalent condition for potential singular solutions

In the Introduction the fundamental, unsolved problem of smooth solutions of the three-dimensional Euler equations developing singularities in a finite time was discussed along with certain criteria for the growth of vorticity. In this section, an equivalent condition to that of a vorticity constraint will be derived. All these additional conditions provide checks for any computational numerical research on any potential singular solutions. In the Appendix these constraints will be considered when the $\alpha$ and $\chi$ variables are modelled using data from the Met Office Unified Model.

On a three-dimensional periodic domain $\Omega$, an $L^m$-norm is defined as

$$
\|\omega(\cdot, t)\|_m \equiv \left[ \int_{\Omega} |\omega(x, t)|^m dV \right]^{1/m}, \quad 1 \leq m < \infty, \quad (3.15)
$$

$$
\|\omega(\cdot, t)\|_\infty \equiv \operatorname{ess sup}_{x \in \Omega} |\omega(x, t)|, \quad m = \infty. \quad (3.16)
$$

Recall from (2.6) that the magnitude of vorticity $|\omega|$ satisfies the equation

$$
\frac{\partial |\omega|}{\partial t} + u \cdot \nabla |\omega| = \alpha |\omega|, \quad (3.17)
$$

therefore the following integral can be simplified as follows

$$
\frac{d}{dt} \int_{\Omega} |\omega|^m dV = m \int_{\Omega} |\omega|^{m-1} (\alpha |\omega| - u \cdot \nabla |\omega|) dV;
$$

$$
= m \int_{\Omega} \alpha |\omega|^m dV - \int_{\Omega} u \cdot \nabla |\omega|^m dV. \quad (3.18)
$$

The second term in (3.18) can be re-written (see glossary) as

$$
\int_{\Omega} u \cdot \nabla |\omega|^m dV = \int_{\Omega} \nabla \cdot (|\omega|^m u) dV - \int_{\Omega} |\omega|^m \nabla \cdot u dV, \quad (3.19)
$$

the second term on the right-hand side is zero due to the flow being non-divergent and using the divergence theorem (see glossary) the expression in (3.18) can be simplified to
\[
\frac{d}{dt} \int_{\Omega} |\omega|^m dV = m \int_{\Omega} \alpha |\omega|^m dV - \int_{\partial\Omega} |\omega|^m \mathbf{u} \cdot \mathbf{n} \ d(\partial V).
\] (3.20)

The surface integral in (3.20) is also zero due to the boundary condition \( \mathbf{u} \cdot \mathbf{n} = 0 \) on \( \partial \Omega \) therefore

\[
\frac{d}{dt} \int_{\Omega} |\omega|^m dV = m \int_{\Omega} \alpha |\omega|^m dV.
\] (3.21)

This result further implies that the following inequality holds

\[
\frac{d}{dt} \int_{\Omega} |\omega|^m dV \leq m \| \alpha \|_{\infty} \int_{\Omega} |\omega|^m dV,
\]

therefore

\[
\frac{d}{dt} \| \omega \|_m^m \leq m \| \alpha \|_{\infty} \| \omega \|_m^m,
\] (3.22)

taking the derivative of the left-hand side and simplifying to give

\[
\frac{d}{dt} \| \omega (\cdot, t) \|_m \leq \| \alpha (\cdot, t) \|_{\infty} \| \omega (\cdot, t) \|_m,
\] (3.23)

finally, integrating this equation gives a further condition for the development of singular solutions to the Euler equations in finite time in terms of the \( L^\infty \) norm of the stretching rate

\[
\| \omega (\cdot, t) \|_m \leq \omega_0 |m \exp \int_0^t \| \alpha (\cdot, \tau) \|_{\infty} d\tau,
\] (3.24)

where \( \omega_0 |m \) is the \( L^m \)-norm of the initial vorticity. The result (3.24) also holds for the limit \( m \to \infty \). Later in this chapter, further constraints on the development of singular solutions will be presented in terms of key quantities present in the evolution equations for the \( \alpha \) and \( \chi \) variables. However, the mathematical techniques to explicitly calculate these prognostic equations for \( (\alpha, \chi) \) must now be derived.
3.4 The evolution of the vorticity stretching rate and pressure Hessian

The explicit form of the vorticity for the Euler equations is now considered to calculate the evolution of the vorticity stretching vector. Consider the derivative

$$\frac{D}{Dt} (\omega \cdot \nabla \mu),$$

for an arbitrary scalar $\mu$. Now consider the vortex stretching written in suffix notation $\omega \cdot \nabla \mu = \omega_i \mu_{,i}$ therefore

$$\frac{D}{Dt} (\omega_i \mu_{,i}) = \frac{D\omega_i}{Dt} \mu_{,i} + \omega_i \frac{D\mu_{,i}}{Dt}. \quad \text{(3.25)}$$

Consider the vector gradient of the evolution equation for the scalar $\mu$

$$\frac{\partial}{\partial x_i} \left( \frac{D\mu}{Dt} \right) = \frac{\partial}{\partial x_i} (\mu_t + u_j \mu_{,j}),$$

$$= \mu_{t,i} + u_j \mu_{,ji} + u_j \mu_{,j},$$

$$= \frac{D}{Dt} (\mu_{,i}) + u_j \mu_{,j}. \quad \text{(3.26)}$$

Substituting equation (3.26) into (3.25) gives

$$\frac{D}{Dt} (\omega_i \mu_{,i}) = \frac{D\omega_i}{Dt} \mu_{,i} + \omega_i \left\{ \frac{\partial}{\partial x_i} \left( \frac{D\mu}{Dt} \right) - u_j \mu_{,j} \right\},$$

$$= \omega_k u_{i,k} \mu_{,i} + \omega_i \frac{\partial}{\partial x_i} \left( \frac{D\mu}{Dt} \right) - \omega_i u_j \mu_{,j}. $$

Summing over the $i, j, k$ variables it is clear that the first and third terms are identical and so if $\omega$ evolves according to the vorticity equation (3.8) then any arbitrary scalar $\mu$ satisfies

$$\frac{D}{Dt} (\omega \cdot \nabla \mu) = \omega \cdot \nabla \left( \frac{D\mu}{Dt} \right). \quad \text{(3.27)}$$

This result, generally credited to Ertel (1942) and known as Ertel’s theorem, says that if the vorticity evolves according to equation (3.8) then any arbitrary scalar
Chapter 3. The inertial incompressible Euler equations

satisfies equation (3.27). This result has been used widely in geophysical fluid dynamics in the study of potential vorticity see Hide (1983) and Hoskins et al. (1985). In a more general sense it can be applied to any fluid system whose flow preserves a scalar or, in fact, a vector field. In this case the Euler equations preserve the vorticity stretching operator \((\omega \cdot \nabla)\). The history of this Ertel result, which seems to have originated in the work of Cauchy, can be seen in Truesdell and Toupin (1960), Kuznetsov and Zakharov (1997) and Viudez (1999).

Substituting \(\mu = u_i\), the \(i\)-th component of the velocity field, into (3.27) gives

\[
\frac{D}{Dt} (\omega \cdot \nabla u_i) = \omega \cdot \nabla \left( \frac{D u_i}{Dt} \right),
\]

\[
= -\omega_j \frac{\partial}{\partial x_j} \left( \frac{\partial p}{\partial x_i} \right),
\]

\[
= -\omega_j p_{,ij}. \tag{3.28}
\]

So, in its most general form, Ertel’s theorem says that if the vorticity satisfies (3.8) then for some arbitrary differentiable vector \(\mu\)

\[
\frac{D}{Dt} (\omega \cdot \nabla \mu) = \omega \cdot \nabla \left( \frac{D \mu}{Dt} \right), \tag{3.29}
\]

and letting \(\mu\) be equal to the velocity field \(u\) then the evolution of the vorticity stretching vector is given by

\[
\frac{D \sigma}{Dt} = \frac{D}{Dt} (\omega \cdot \nabla u) = -P \omega, \tag{3.30}
\]

where \(P = \{p_{,ij}\}\) is the Hessian matrix of the pressure given explicitly by

\[
P = \begin{pmatrix}
\frac{\partial^2 p}{\partial x^2} & \frac{\partial^2 p}{\partial x \partial y} & \frac{\partial^2 p}{\partial x \partial z} \\
\frac{\partial^2 p}{\partial y \partial x} & \frac{\partial^2 p}{\partial y^2} & \frac{\partial^2 p}{\partial y \partial z} \\
\frac{\partial^2 p}{\partial z \partial x} & \frac{\partial^2 p}{\partial z \partial y} & \frac{\partial^2 p}{\partial z^2}
\end{pmatrix}. \tag{3.31}
\]

There are a number of consequences that the relation in (3.30) highlights and they are worth considering in more detail. By taking the material derivative of the vorticity equation (3.8) and applying the result to the evolution of the vorticity stretching
vector (3.30), gives the following relationship between the vorticity and the pressure Hessian that

\[ \frac{D^2 \omega}{Dt^2} + P \omega = 0. \]  \hspace{1cm} (3.32)

This equation is known as the Ohkitani relation, Ohkitani (1993). Although it may look particularly straightforward, this is misleading, as the second derivative of the vorticity is a material one. It is stated in Galanti et al. (1997), that at or near alignment of the vorticity \( \omega \) with an eigenvector of \( P \) it is the negative eigenvalues of \( P \) that cause exponential growth in the vorticity. Secondly, the vital step in the derivation of (3.27), in which the two terms of order \( \omega |\nabla u \cdot \nabla \mu| \), or more specifically \( \omega |\nabla u| \) when \( \mu = u_i \), are cancelled, removes all the non-pressure dependent terms. This derivation is in contrast to the usual process of the pressure being removed by projection or by considering the corresponding vorticity equation. The advantage of this formulation is that it removes the non-linear vortex stretching terms. This removal of non-linear terms comes with the added complication of having to consider the pressure Hessian matrix.

### 3.5 The quaternionic structure in the incompressible Euler equations

The explicit forms that the evolution equations for the stretching rate and vortex alignment vector take for the incompressible Euler equations (see Galanti et al. (1997)) are found by substituting the result in equation (3.30) into equations (2.18) and (2.19) to give

\[ \frac{D\alpha}{Dt} + \alpha^2 - |\chi|^2 = -\frac{\omega \cdot P \omega}{\omega \cdot \omega}, \]  \hspace{1cm} (3.33)

\[ \frac{D\chi}{Dt} + 2\chi \alpha = -\frac{\omega \times P \omega}{\omega \cdot \omega}. \]  \hspace{1cm} (3.34)
The evolution equations for the vortex stretching and alignment can be re-written in terms of the single equation (2.29) for the 4-vector $\mathbf{q} = (\alpha, \chi)^T$ where $\mathbf{s}_1$ is given by $(0, -P\mathbf{\omega})^T$. The equation dependent term in (2.29) can be re-written as the single 4-vector $\mathbf{q}_p = (\alpha_p, \chi_p)$ where

$$\alpha_p = \frac{\mathbf{\omega} \cdot P\mathbf{\omega}}{\mathbf{\omega} \cdot \mathbf{\omega}}, \quad \chi_p = \frac{\mathbf{\omega} \times P\mathbf{\omega}}{\mathbf{\omega} \cdot \mathbf{\omega}}. \quad (3.35)$$

These expressions for $\alpha_p$ and $\chi_p$ are identical to equations (3.10) and (3.11) respectively with the pressure Hessian matrix $P$ replacing the strain matrix $S$.

The quaternionic Riccati equation for the evolution of the 4-vector $\mathbf{q}$ in terms of the dependent variables $\mathbf{q}$ and $\mathbf{q}_p$ is

$$\frac{D\mathbf{q}}{Dt} + \mathbf{q} \otimes \mathbf{q} = -\mathbf{q}_p. \quad (3.36)$$

It is now possible to give further meaning for the $\chi$ vector in the context of the 4-vector $\mathbf{q}$. Turbulent vorticity fields are well known to be dominated by vortex sheet-like and tube-like features - see Vincent and Meneguzzi (1994), Kerr (1993) and Frisch (1995). For straight (vortex) tubes or flat (vortex) sheets, the vorticity aligns itself with an eigenvector of the strain matrix $S$, and so the spin rate, $\chi$ is zero and so the stretching rate is an exact eigenvalue of the strain matrix. This is an example when the alignment angle $\phi$ is zero - if the angle where $\pi$ then we have anti-alignment. In these cases the 4-vector Riccati equation for $\mathbf{q}$ reduces to a scalar equation for the stretching rate given by $\mathbf{q} = \alpha \mathbf{1}$ and the system reduces to a simple problem in the stretching rate alone (Vieillefosse (1984)). However, for vortex tubes that twist or bend, then $\chi \neq 0$ and the full 4-vector equation for $\mathbf{q}$ is restored. The tendency for the vorticity to align with certain eigenvectors of the strain matrix, known as preferential alignment, has been of the main themes of computational research in both inviscid and viscous turbulence in recent years (see Ashurst et al. (1987), Jimenez (1992) and Tsinober et al. (1992)).

Let us consider these results for the particular example of Burgers’ solutions to the Euler equations.
3.5.1 Burgers’ solutions to the Euler equations

A set of solutions to the Euler equations are given by the velocity field \( \mathbf{u} \)

\[
\mathbf{u} = \left( -\frac{\delta}{2}x, -\frac{\delta}{2}y, \delta z \right) + \left( \Phi_y, \Phi_x, 0 \right),
\]

where \( \delta \) is constant and the vorticity is in the vertical direction given by the Laplacian of \( \Phi \). The strain matrix is the block diagonal form

\[
S = \begin{pmatrix}
-\frac{\delta}{2} - \Phi_{yy} & \frac{1}{2} (-\Phi_{yy} + \Phi_{xx}) & 0 \\
\frac{1}{2} (\Phi_{xx} - \Phi_{yy}) & -\frac{\delta}{2} + \Phi_{yx} & 0 \\
0 & 0 & \delta
\end{pmatrix}.
\]

(3.38)

One eigenvector of \( S \) is \((0, 0, 1)^T\), and so combined with the result that the vorticity is strictly in the vertical direction, then the vorticity is parallel to this eigenvector and so \( \chi = 0 \). Our four vorticity and pressure variables are given by

\[
\alpha = \delta, \quad \chi = 0, \\
\alpha_p = -\delta^2, \quad \chi_p = 0,
\]

(3.39)

and furthermore

\[
\frac{D\alpha}{Dt} = \frac{D\chi}{Dt} = 0;
\]

(3.40)

hence the Burgers’ solutions are like "equilibrium solutions" of the system in the Lagrangian sense although the solutions are not necessarily stable. The solutions for the scalar vorticity \( \omega \), in terms of \( r^2 = x^2 + y^2 \), corresponding to the axisymmetric Burgers’ vortex, is given by

\[
\omega(r,t) = \exp(\delta t) \omega_0 \left( r \exp \left( \frac{\delta t}{2} \right) \right),
\]

(3.41)

where \( \omega_0 \) is the initial vorticity \( \omega_0 = \omega(r,0) \). For \( \delta > 0 \) the support collapses exponentially as the amplitude grows producing an ever-thinning vortex tube. If \( \delta < 0 \) then the exact opposite occurs and the result is an ever-flattening vortex
sheet. In terms of the 4-vector \( q \), due to the alignment of the vorticity with one of the eigenvectors of \( S \), \( \chi = 0 \), and so

\[
q = \delta (1, 0)^T, \quad q_p = -\delta^2 (1, 0)^T,
\]

and \( q \) reduces to a scalar equation for the stretching rate \( \alpha \) given by \( q = \alpha \mathbf{1} \).

### 3.6 The complex structure in the incompressible Euler equations

Returning to the 4-vector equation for the evolution of \( q \) it is possible to reduce the quaternionic Riccati equation for \( q \) to a complex Riccati equation by following the procedure outlined in section 2.3. The corresponding equation for the scalar \( |\chi| \) is

\[
\frac{D|\chi|}{Dt} = -2|\chi|\alpha + |\omega|^{-2} \hat{\chi} \cdot (\omega \times -P\omega),
\]

which can be simplified using the expression for \( \chi_p \) given in (3.35) to

\[
\frac{D|\chi|}{Dt} = -2|\chi|\alpha - \hat{\chi} \cdot \chi_p.
\]

The complex Riccati equation in terms of \( \zeta_c \) is

\[
D\zeta_c + \zeta_c^2 + \psi_c = 0.
\]

Linearising this equation using the substitution (2.39) gives

\[
D^2 \gamma_c + \psi_c \gamma_c = 0;
\]

this is a zero eigenvalue (scalar) Schrodinger equation for the transformed variable \( \gamma_c \) with potential \(-\psi_c\). This differs only slightly from the expression in (2.40)
due to the negative sign in the expression for the evolution of the vortex stretching term. Before the work of Adler and Moser (1978) on the infinite set of solutions of scalar zero-eigenvalue Schrödinger equations and the corresponding work of Gibbon (2002) for the current complex form of the problem is discussed, the constraint equation for the Euler equation shall be derived.

### 3.7 The constraint equation for the Euler equations

To “close” the system, due to the addition of an extra prognostic equation defining the motion, an equation providing information on any relationships between these dependent variables, namely, the vorticity and the strain and pressure Hessian matrices, needs to be derived. This constraint equation will also provide information regarding any relationships between the two 4-vectors $\mathbf{q}$ and $\mathbf{q}_p$ seen in the evolution equation for $\mathbf{q}$. Consider the momentum equation (3.4) re-written using suffix notation

$$u_{i,t} + u_j u_{i,j} = -p_{,i}.$$  \hfill (3.48)

Take the divergence of equation (3.48), and note that $u_{i,i} = 0$ from the incompressibility condition (3.3), to give

$$u_{i,ti} + \frac{\partial}{\partial x_i} (u_j u_{i,j}) = -p_{,ii},$$

$$u_{i,ti} + u_{j,i} u_{i,j} + u_j u_{i,ji} = -p_{,ii},$$

the first and third terms cancel to give

$$u_{j,i} u_{i,j} = -p_{,ii}.$$  \hfill (3.49)

Taking the divergence of the momentum equation produces the Poisson equation

$$\Delta p = -u_{j,i} u_{i,j}.$$
where \( \Delta \) is the three-dimensional Laplacian operator \( \partial^2 / \partial x_i \partial x_i \). This equation can also be expressed in terms of the pressure Hessian \( P \), the strain matrix \( S \) and the vorticity \( \omega \) by first noting that the Laplacian of the pressure is the trace of the pressure Hessian matrix, \( \Delta p = \text{Tr} P \) and, secondly, by noting that, although the trace of the strain matrix is zero from the incompressibility condition, the trace of the square of strain matrix is given by

\[
\text{Tr} S^2 = \left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial v}{\partial y} \right)^2 + \left( \frac{\partial w}{\partial z} \right)^2 + \frac{1}{2} \left\{ \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right)^2 + \left( \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right)^2 + \left( \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right)^2 \right\}.
\]

Furthermore, the right-hand side of this equation, when written out in full is given by

\[
u_{i,j} u_{j,i} = \left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial v}{\partial y} \right)^2 + \left( \frac{\partial w}{\partial z} \right)^2 + 2 \left( \frac{\partial u}{\partial y} \frac{\partial v}{\partial x} + \frac{\partial u}{\partial z} \frac{\partial w}{\partial x} + \frac{\partial v}{\partial z} \frac{\partial w}{\partial y} \right),
\]

and finally when \(|\omega|^2\) is expanded, \( u_{i,j} u_{j,i} = \text{Tr} S^2 - \frac{1}{2} |\omega|^2 \), and so the complete constraint equation analogous to (3.49) is given by

\[
\text{Tr} P = \frac{1}{2} |\omega|^2 - \text{Tr} S^2.
\] (3.50)

These two equations (3.49) and (3.50) provide an explicit relationship between the dependent variables, and also give us a greater insight into (3.36), namely that the 4-vector \( q_p \) is not completely independent of \( q \).

### 3.8 The work of Adler & Moser and the complex Schrödinger equation

Recall that the complex zero-eigenvalue Schrödinger equation (3.47) is

\[
\frac{D^2 \gamma_c}{Dt^2} + \psi_c \gamma_c = 0,
\] (3.51)
where $\psi_c = \alpha_p + i \hat{\chi} \cdot \chi_p$ and $\zeta_c = \gamma_c^{-1} \frac{D\gamma_c}{Dt} = \alpha + i|\chi|$. Consider the problem in a corresponding Lagrangian framework where the basis of the co-ordinate system is defined using particle labels $a = (a, b, c)$ and let these labels take the form of the (Eulerian) position of the fluid at some chosen (or initial) time $\tau$. Therefore the position vector $x = x(a, t)$, and the first component of the velocity field is

$$u(a, t) = \frac{\partial x}{\partial t}(a, t) = \dot{x}(a, t),$$

(3.52)

with similar expressions for the $v$ and $w$ components. Within this framework, the Schrödinger equation (3.51) becomes

$$-\dddot{\gamma} + U\gamma = 0,$$

(3.53)

with the potential $U = -\psi_c(a, t)$ and the double dot is two Lagrangian time derivatives (i.e. $\partial^2 / \partial t^2$). The aim is that given a complex potential $U(t)$ a solution is sought for (3.53) that would enable one to find $\zeta_c$ and hence its real and imaginary parts, which would be $\alpha$ and $\chi$. However, solving this does not help with the greater problem of determining the fluid particle trajectories that would correspond to this outlined solution. From the previous section on the constraint equation it was made clear that the Hessian matrix of the pressure was not an independent quantity and in fact relies on the strain matrix and the vorticity (3.50). Therefore, the particle paths of the flow must be compatible with this constraint and the answer to this problem is not, as yet, known. Although solving (3.53) is only part of our problem it is worth considering its solutions as they are, in their own right, quite interesting. Adler and Moser proved the following theorem for the zero-eigenvalue Schrödinger problem

**Theorem** (Adler and Moser (1978)) For potentials $U(t)$ in (3.53) that take the form

$$U_k = -2 \frac{\partial^2}{\partial t^2} \ln \theta_k,$$

(3.54)

the eigenfunctions $\gamma_k$ satisfy

$$\gamma_k = \theta_{k+1} \theta_k^{-1},$$

(3.55)
where the infinite set of polynomials $\theta_k$ of degree $n_k = \frac{1}{2} k (k + 1)$ can be generated from the nonlinear Wronskian recurrence relation

$$\dot{\theta}_{k+1}\theta_{k-1} - \theta_{k+1}\dot{\theta}_{k-1} = (2k + 1) \theta_{k}^2,$$

starting from $\theta_0(t) = 1$, $\theta_1(t) = t + c_1$.

The proof of this theorem is given in detail in Adler and Moser (1978) for potentials $U \in \mathbb{R}$. This is expanded in Gibbon (2002) for cases in which $U \in \mathbb{C}$. This recurrence relationship (3.56) will generate $\theta_k$ to any desired order. The first four are given by

$$\begin{align*}
\theta_0(t) &= 1, \\
\theta_1(t) &= t + c_1, \\
\theta_2(t) &= t^3 + 3t^2 c_1 + 3tc_1^2 + c_2, \\
\theta_3(t) &= 5(t + c_1) \int t \frac{\theta_2(t')}{(t' + c_1)^2} dt',
\end{align*}$$

where the $c_k$ are arbitrary (complex) constants. Furthermore, Adler and Moser (1978) have shown that a generating function exists for these polynomials. Finally, $\zeta_c$, the complex form of $q$, can be expressed (with a $k$ suffix) as

$$\begin{align*}
(\zeta_c)_k = \frac{\partial}{\partial \tau} (\ln \gamma_k) = \frac{\partial}{\partial \tau} \ln \left( \frac{\theta_{k+1}}{\theta_k} \right),
\end{align*}$$

and the real and imaginary parts of (3.61) give $\alpha_k$ and $\chi_k$ respectively. These solutions mean that the $\gamma_k$ expressed through the $\theta_k$ correspond to the class of potentials $U_k$ that were initially given in equation (3.53). It is the particle paths that relate to or correspond to equations (3.54) and (3.55) that must be consistent with the constraint equation for the pressure Hessian (3.50) and, finally, the complex constants, $c_k$, would have to be calculated in term of the particle path positions. As was explicitly mentioned earlier, this compatibility between the particle paths and the constraint equation is not as yet known. One final note regarding singularities in these solutions as mentioned in Gibbon (2002), is that as the $\tau_k$ are complex constants, any singular solutions must lie off the real axis unless, of course, the $\tau_k$ are
Chapter 3. The inertial incompressible Euler equations

3.9 Equivalent conditions for potential singular solutions II

Returning to the early problem of the development of singular solutions to the Euler equations in finite time by using the evolution equations for the stretching rate and vortex alignment vector, further criteria for the development of singular solutions in terms of key quantities can be developed. Recall that the evolution equations for \( \alpha \) and \( \chi \) are given by

\[
\frac{D\alpha}{Dt} = |\chi|^2 - \alpha^2 - \alpha_p, \quad \frac{D\chi}{Dt} = -2\chi\alpha - \chi_p,
\]  

where the \((\alpha_p, \chi_p)\) variables are defined in equation (3.35). Using these evolution equations for \((\alpha, \chi)\) give

\[
\frac{D}{Dt} \left[ \frac{1}{2} (\alpha^2 + |\chi|^2) \right] = \alpha (|\chi|^2 - \alpha^2 - \alpha_p) - 2\alpha|\chi|^2 - \chi \cdot \chi_p, \]

\[
= -\alpha (\alpha^2 + |\chi|^2) - \alpha \alpha_p - \chi \cdot \chi_p. \]

Defining new variables

\[
X^2 = \alpha^2 + |\chi|^2, \quad X_p^2 = \alpha_p^2 + |\chi_p|^2,
\]

and recalling that if the stretching rate is negative then this leads to vortex compression, we find that the vorticity collapses. Thus for positive values of the stretching rate the following inequality holds

\[
X \frac{DX}{Dt} \leq -\alpha \alpha_p - \chi \cdot \chi_p \leq XX_p + XX_p = 2XX_p. \]
Chapter 3. The inertial incompressible Euler equations

From this the following bounds for the $L^\infty$-norm stretching rate in terms of these new variables $X, X_p$ can be written as

$$\|\alpha (\cdot, t)\|_\infty \leq \|X (\cdot, t)\|_\infty \leq 2 \int_0^t \|X_p (\cdot, \tau)\|_\infty d\tau. \quad (3.66)$$

Finally, combining this with the result given in equation (3.24) for the stretching rate we see that the $L^\infty$-norm of the vorticity is controlled by the double integral

$$\int_0^t \int_0^{t'} \|X_p (\cdot, \tau)\|_\infty d\tau d\tau'. \quad (3.67)$$

There are two important things to note from this, first, $\|X_p (\cdot, t)\|_\infty$ is bounded by the maximum eigenvalue of the pressure Hessian matrix and secondly, a consequence of this result is that the $L^\infty$ vortex stretching rate is therefore bounded by the maximum row sum of the pressure Hessian matrix $P$.

### 3.10 Quaternionic form of the momentum equation

In the previous chapter the quaternionic structure was derived from a general vorticity equation without any mention of the particular form of the corresponding momentum equation. However, as has been shown, the momentum equation is vital in first, deriving the explicit form of the vorticity equation and secondly, in closing the problem by providing the constraint equation that gives a natural relationship between the dependent variables. Let’s reconsider the momentum equation (3.1) and incompressibility constraint (3.3) and attempt to write them in terms of the quaternionic algebra seen in the previous chapter. By bringing the momentum equation into such a framework then an interesting result is seen when the quaternionic ”curl” is taken. Defining the following 4-vectors

$$\mathbf{U} = (0, \mathbf{u})^T, \quad \mathbf{P} = (p, 0)^T, \quad \nabla = (0, \nabla)^T,$$

the quaternionic form of the Euler momentum equation, in Lagrangian form, in terms of $\mathbf{U}$ and $\mathbf{P}$ is
The quaternionic vorticity vector $\mathbf{w}$ can be formed by considering the quaternionic curl operator applied to the 4-vector $\mathbf{U}$

$$
\nabla \otimes \mathbf{U} = (0, \nabla)^T \otimes (0, \mathbf{u})^T = (-\nabla \cdot \mathbf{u}, \nabla \times \mathbf{u})^T = (0, \omega)^T = \mathbf{w}.
$$

(3.70)

To remove the material derivative operator from equation (3.69) recall that the alternative form of the momentum equation

$$
\frac{\partial \mathbf{u}}{\partial t} + (\nabla \times \mathbf{u}) \times \mathbf{u} + \frac{1}{2} \nabla|\mathbf{u}|^2 = -\nabla p,
$$
can be expressed in 4-vector form as (and dropping the $T$-transpose notation)

$$
\begin{align*}
(0, \frac{\partial \mathbf{u}}{\partial t}) &= (0, \mathbf{u} \times \omega) - \left(0, \nabla \left\{ p + \frac{1}{2} \mathbf{u}^2 \right\} \right), \\
(0, \frac{\partial \mathbf{u}}{\partial t}) &= \frac{1}{2} \{ (-\mathbf{u} \cdot \omega, \mathbf{u} \times \omega) - (-\omega \cdot \mathbf{u}, \omega \times \mathbf{u}) \} - \left(0, \nabla \left\{ p + \frac{1}{2} \mathbf{u}^2 \right\} \right).
\end{align*}
$$

Therefore

$$
\frac{\partial}{\partial t} (0, \mathbf{u}) = \frac{1}{2} \{(0, \mathbf{u}) \otimes (0, \omega) - (0, \omega) \otimes (0, \mathbf{u}) \} - (0, \nabla) \otimes \left( p + \frac{1}{2} \mathbf{u}^2, 0 \right).
$$

(3.72)

Defining the 4-vector $\mathbf{P} = (p + \frac{1}{2} \mathbf{u}^2, 0)$, the fully expanded form of (3.69) is given by

$$
\frac{\partial \mathbf{U}}{\partial t} = \frac{1}{2} \{ \mathbf{U} \otimes \mathbf{w} - \mathbf{w} \otimes \mathbf{U} \} - \nabla \otimes \mathbf{P}.
$$

(3.73)

Applying the quaternionic curl operator to the momentum equation in the form given in (3.71) gives

$$
(0, \nabla) \otimes \left\{ \left(0, \frac{\partial \mathbf{u}}{\partial t}\right) - (0, \mathbf{u} \times \omega) + \left(0, \nabla \left\{ p + \frac{1}{2} \mathbf{u}^2 \right\} \right) \right\} = 0.
$$

(3.74)
expanding this gives

\[
\left(-\nabla \cdot \frac{\partial \mathbf{u}}{\partial t}, \nabla \times \frac{\partial \mathbf{u}}{\partial t}\right) = \left(-\nabla \cdot \{\mathbf{u} \times \omega\}, \nabla \times \{\mathbf{u} \times \omega\}\right) \quad (3.75)
\]

\[
- \left(-\nabla \cdot \nabla \left\{ p + \frac{1}{2} \mathbf{u}^2 \right\}, \nabla \times \nabla \left\{ p + \frac{1}{2} \mathbf{u}^2 \right\} \right).
\]

The scalar term on the left-hand side is zero due to the incompressibility constraint and the final 3-vector term on the right-hand side is zero, therefore

\[
\left(0, \frac{\partial}{\partial t} \{\nabla \times \mathbf{u}\}\right) = \left(-\nabla \cdot \{\mathbf{u} \times \omega\}, \nabla \times \{\mathbf{u} \times \omega\}\right) + \left(\Delta \left\{ p + \frac{1}{2} \mathbf{u}^2 \right\}, 0\right).
\]

(3.76)

After expanding and re-arranging this expression

\[
\left(-u_i,j u_{j,i}, \frac{D\omega}{Dt}\right) = (\Delta p, (\omega \cdot \nabla) \mathbf{u}).
\]

(3.77)

So, taking the quaternionic curl of the momentum equation in its 4-vector form not only produces the corresponding vorticity form of the Euler equations but also the constraint equation derived in (3.49). Now, re-consider the evolution equation for a general 4-vector \( \mathbf{q} \) given in terms of a corresponding general vorticity \( \mathbf{w} \), seen in equation (2.35), in light of the quaternionic relationship derived in this chapter for the Euler equations (3.36). In a 4-vector formulation the Ohkitani relationship should take the form

\[
\frac{D^2 \mathbf{w}}{Dt^2} = -\mathbf{q}_p \otimes \mathbf{w}.
\]

(3.78)

**Theorem:** For the three-dimensional incompressible Euler equations the 4-vectors for the vorticity \( \mathbf{w} \) and the pressure Hessian variables, \( \mathbf{q}_p \), are related by the expression in (3.78).

**Proof:** Recall the result that \( S \mathbf{w} = \alpha \mathbf{w} + \chi \times \mathbf{w} \) and so substituting \( S \) for \( P \) then

\[
P \mathbf{w} = \alpha_p \mathbf{w} + \chi_p \times \mathbf{w}.
\]
and the pressure Hessian term \( P\omega \) can be expressed in 4-vector form

\[
(0, P\omega) = (0, \alpha_p \omega + \chi_p \times \omega) = (\alpha_p, \chi_p) \otimes (0, \omega) = q_p \otimes \mathbf{w},
\]

where the orthogonality condition \( \chi_p \cdot \omega = 0 \) has been applied. Considering the second material derivative of the vorticity 4-vector

\[
\frac{D}{Dt} \left( \frac{D\mathbf{w}}{Dt} \right) = \frac{D}{Dt} (0, S\omega) = -(0, P\omega) = -q_p \otimes \mathbf{w},
\]

and so

\[
\frac{D^2\mathbf{w}}{Dt^2} = -q_p \otimes \omega.
\]

Substituting this expression for the second Lagrangian derivative of the vorticity 4-vector into equation (2.35) gives the previous derived quaternionic Riccati equation for \( q \) given by equation (3.36).

### 3.11 Evolution equations for the pressure Hessian variables

This section on defining Lagrangian advection equations for \( \alpha_p \) and \( \chi_p \) can be found in Gibbon et al. (2006). The orthogonality relationship \( \chi_p \cdot \omega = 0 \) mentioned briefly in this chapter, although rather innocent looking, is the main focus of attention for this next section. Previously, the evolution equation derived for the vector \( q \) was in terms of a pressure variable \( q_p \), however, there is no Lagrangian differential equation for \( q_p \) and this is one of the hurdles in pursuing a Lagrangian approach to the study of Euler; the problem of the non-locality of the pressure field. Numerically, the pressure is updated from the previous derived Poisson equation \( \Delta p = -u_{i,j} u_{j,i} \).

However, it is possible to derive prognostic equations for the pressure variables by considering a variation of the orthogonality relationship, that is, by replacing the vorticity vector with the vorticity unit vector. Then
\( \chi_p \cdot \hat{\omega} = 0 \Rightarrow \frac{D \chi_p}{Dt} \cdot \hat{\omega} + \chi_p \cdot \frac{D \hat{\omega}}{Dt} = 0. \) \hspace{1cm} (3.79)

Substituting the result for the evolution of the vorticity unit vector (2.15) into (3.79) gives

\[ \hat{\omega} \cdot \left( \frac{D \chi_p}{Dt} + \chi_p \times \chi \right) = 0, \] \hspace{1cm} (3.80)

which implies that

\[ \frac{D \chi_p}{Dt} = \chi \times \chi_p + q, \] \hspace{1cm} (3.81)

where \( \mu(x,t), \lambda(x,t) \) are unknown scalars. For \( \alpha_p \) we need an explicit expression for the derivative of \( P\hat{\omega} \) as \( \alpha_p = \hat{\omega} \times P\hat{\omega} \). The corresponding expression for \( \chi_p \) in terms of the unit vorticity vector is given by \( \chi_p = \hat{\omega} \times P\hat{\omega} \) and so

\[ \frac{D \chi_p}{Dt} = \frac{D \hat{\omega}}{Dt} \times P\hat{\omega} + \hat{\omega} \times \frac{D}{Dt} (P\hat{\omega}). \] \hspace{1cm} (3.82)

Substituting equation (3.81) into the above gives

\[ \hat{\omega} \times \frac{D}{Dt} (P\hat{\omega}) = q + \alpha_p \chi, \] and re-arranging and simplifying this expression gives

\[ \frac{D}{Dt} (P\hat{\omega}) - \alpha_p \omega = q \times \hat{\omega} + \epsilon \hat{\omega}, \]

where \( \epsilon(x,t) \) is a third, unknown scalar. So the expression for the evolution of the pressure stretching rate variable \( \alpha_p \) is given by

\[ \frac{D\alpha_p}{Dt} = \frac{D}{Dt} (\hat{\omega} \cdot P\hat{\omega}) = \alpha \alpha_p + \chi \cdot \chi_p + \epsilon. \] \hspace{1cm} (3.83)

Althought Lagrangian derivatives have been found for \((\alpha_p, \chi_p)\) they are at the expense of introducing three new scalars. These scalars are not arbitrary and will need to be adjusted in a flow to take the corresponding Poisson constraint into account.
3.11.1 A quaternionic representation of the pressure 4-vector $q_p$

The obvious question is can these evolution equations be re-expressed as a single equation for the 4-vector $q_p$? Considering the product $q \otimes q_p$ then

$$\frac{D q_p}{Dt} = q \otimes q_p + \left( \text{epsilon} + 2 \chi \cdot \chi_p \right) \left( (\mu - \alpha_p) \chi + (\lambda - \alpha) \chi_p \right),$$

Introducing three new scalars as

$$\lambda_1 = \lambda - \alpha, \mu_1 = \mu - \alpha_p, \epsilon_1 = \epsilon = 2 \chi \cdot \chi_p - \mu_1 \alpha - \lambda_1 \alpha_p. \quad (3.84)$$

This re-labelling is permissible as a dimensional analysis of the $(\alpha, \chi)$ and $(\alpha_p, \chi_p)$ variables, that prescribe the scalings of the $(\lambda, \mu, \epsilon)$ scalars, are consistent with the definitions of these new variables seen in equation (3.84). Defining a new 4-vector

$$q_{\mu,\lambda,\epsilon} = \mu_1 q + \lambda_1 q_p + \epsilon_1 1, \quad (3.85)$$

then the pressure 4-vector $q_p$ satisfies

$$\frac{D q_p}{Dt} = q \otimes q_p + q_{\mu,\lambda,\epsilon}, \quad (3.86)$$

where the scalars in $q_{\mu,\lambda,\epsilon}$ are determined by the Poisson equation constraint. One question is what effect does the Poisson equation have on the scalars in $q_{\mu,\lambda,\epsilon}$? This is not as yet known but by considering the earlier example of Burgers vortex they are not all likely to be zero, as $\alpha = \delta, \alpha_p = -\delta^2, \chi = \chi_p = 0$ and so

$$\lambda_1 = \mu_1 = 0 \quad \text{but} \quad \epsilon_1 = \delta^3.$$

3.12 Comparison analysis with the Navier-Stokes equations

Before a detailed analysis of certain approximations are considered it will be informative to consider the Euler equations with viscosity - the Navier-Stokes equations
Chapter 3. The inertial incompressible Euler equations

- in two different ways. The first, using the theory set out in Chapter 2 and also by the classic approach set out in Galanti et al. (1997). The relative advantages and disadvantages of both methods will be discussed.

The momentum equation for the flow of an ideal, viscous fluid is given by

\[ \frac{D\mathbf{u}}{Dt} = -\frac{1}{\rho} \nabla p + \nu \Delta \mathbf{u}, \]  

(3.87)

where \( \nu \) is the coefficient of viscosity. Together with the incompressible constraint (3.3) gives the Navier-Stokes equations. The corresponding vorticity equation is given by

\[ \frac{D\omega}{Dt} = (\omega \cdot \nabla) \mathbf{u} + \nu \Delta \omega. \]  

(3.88)

The first analysis considered is the approach set out in Galanti et al. (1997), and is discussed below.

3.12.1 The classical approach

The vorticity equation (3.88) is re-written as

\[ \frac{D\omega}{Dt} = \sigma + \nu \Delta \omega, \]  

(3.89)

where \( \sigma \) is strictly the vorticity stretching \((\omega \cdot \nabla) \mathbf{u}\). The scalar \( \alpha \) and the 3-vector \( \chi \) are still defined as in equations (2.7) and (2.8) respectively. However, the \( \alpha \) scalar is no longer defined as in equation (2.6) and the evolution of the vortex magnitude is given by

\[ \frac{D|\omega|}{Dt} = \alpha |\omega| + \nu \hat{\omega} \cdot \Delta \omega, \]  

(3.90)

and \( \alpha \) is no longer the vortex stretching rate. However with this particular choice of \( \sigma \), the \( \alpha \) and \( \chi \) variables are given explicitly by (3.10) and (3.11), and these variables still provide information, along with the local angle \( \phi \), of the alignment of the vorticity with \( S \omega \), or more explicitly the eigenvectors of \( S \). To find the
evolution equations for $\alpha$ and $\chi$ the result (3.27) is no longer valid as the equation for the vorticity is no longer given by (3.8). In fact, the evolution equations for these variables derived from the Navier-Stokes equations are

$$\frac{D\alpha}{Dt} = |\chi|^2 - \alpha^2 + \nu\Delta\alpha + 2\nu\alpha|\nabla\hat{\omega}|^2 + \lambda,$$  
(3.91)

$$\frac{D\chi}{Dt} = -2\alpha\chi + \nu\Delta\chi + 2\nu\chi|\nabla\hat{\omega}|^2 + \mu.$$  
(3.92)

where $\lambda, \mu$ are quite complicated variables based on the evolution of the vorticity unit vector for the Navier-Stokes equations. It is possible to combine these two equations using the 4-vector $q$ to give

$$\frac{Dq}{Dt} = -q \otimes q + \nu \left( \Delta q + 2|\nabla\hat{\omega}|^2 q \right) + q_{\lambda, \mu},$$  
(3.93)

where $q_{\lambda, \mu} = (\lambda, \mu)^T$. The constraint equation is unaltered, in both this and the thesis approach, from the Euler equations, as the divergence of the viscosity term is zero.

### 3.12.2 Thesis approach

The thesis approach, which is set out in the previous chapter, says that the vorticity equation takes the form (2.1) where

$$\sigma = (\omega \cdot \nabla) u + \nu\Delta\omega = S\omega + \nu\Delta\omega.$$  

The vorticity stretching vector now incorporates the effect of viscosity. This representation of the vortex stretching vector $\sigma$ means that the equation for the evolution of the vortex magnitude (2.6) holds and therefore $\alpha$ is still the vortex stretching rate and the $(\alpha, \chi)$ variables are given explicitly by

$$\alpha \equiv \frac{\omega \cdot \sigma}{\omega \cdot \omega} = \frac{\omega \cdot S\omega}{\omega \cdot \omega} + \nu \frac{\omega \cdot \Delta\omega}{\omega \cdot \omega},$$

$$\chi \equiv \frac{\omega \times \sigma}{\omega \cdot \omega} = \frac{\omega \times S\omega}{\omega \cdot \omega} + \nu \frac{\omega \times \Delta\omega}{\omega \cdot \omega}.$$  
(3.94)
These variables now provide information between the vorticity, the vorticity stretching vector and the viscosity. To complete the analysis, the material derivative of the vorticity stretching vector has to be derived, and the same problem as with the classical approach still exists, that of the Ertel result (3.27) no longer being valid. In fact a way of calculating

$$\frac{D}{Dt}\{ (\omega \cdot \nabla) u + \nu \Delta \omega \}$$

will be dealt with when a similar problem arises in dealing with a particular type of vorticity equation in a subsequent approximation.

The main differences between these two approaches are the exact forms that the \((\alpha, \chi)\) variables take. In the classic approach these variables are approximately the scalar and vector products of the vorticity and the vorticity stretching and do not incorporate the viscosity. The advantage of such an approach as it allows a direct comparison between these variables for the two sets of governing equations (Euler and Navier-Stokes) under consideration. However, one of the key quantities that is considered in such a comparison is the strain matrix, its corresponding eigenvalues and the alignment of its eigenvectors. What has made this possible is that the vorticity stretching \((\omega \cdot \nabla) u\) can be re-written as \(S \omega\). This, however, is only possible when the vorticity is strictly the curl of the velocity field. Therefore, the incorporation of the strain matrix is no longer practical when the vorticity takes any other, less idealised form, which is the case for all subsequent approximations to the Euler equations. In contrast, the thesis approach incorporates this addition of viscosity into the \((\alpha, \chi)\) variables directly. The most obvious advantage of the thesis approach is that the general results derived in Chapter 2 hold for the Navier-Stokes equations. The reason why the thesis approach is so applicable to this set of equations is that very little is assumed regarding the form that the vorticity equation takes.
Chapter 3. The inertial incompressible Euler equations

3.13 Summary

This chapter has considered the theory of the previous chapter applied to the three-dimensional incompressible Euler and Navier-Stokes equations. The vorticity equation for the Euler equation is given in equation (3.8) and the vorticity stretching vector is expressed in terms of the strain matrix $S$. Explicit forms for the stretching rate and vortex alignment vector are given by

$$\alpha = \frac{\omega \cdot S\omega}{\omega \cdot \omega}, \quad \chi = \frac{\omega \times S\omega}{\omega \cdot \omega},$$

respectively and the role of the local angle is discussed with respect to these variables. The evolution of the vorticity stretching term is calculated using Ertel’s theorem and the evolution equations for $\alpha$ and $\chi$ are given by

$$\frac{D\alpha}{Dt} + \alpha^2 - |\chi|^2 = -\frac{\omega \cdot P\omega}{\omega \cdot \omega},$$

$$\frac{D\chi}{Dt} + 2\chi\alpha = -\frac{\omega \times P\omega}{\omega \cdot \omega}.$$ 

Furthermore, new variables $\alpha_p$ and $\chi_p$ are defined, similar to the expressions for $\alpha$ and $\chi$ with the strain matrix replaced by the pressure Hessian matrix $P$. The quaternionic Riccati equation derived from the incompressible Euler equation is then given by

$$\frac{Dq}{Dt} + q \otimes q = -q_q.$$ 

This equation is then discussed for the case of Burgers’ vortex and furthermore, in terms of flat vortex sheets and the twisting or bending of vortex tubes. This 4-vector/quaternionic equation is then transformed into a 2-vector/complex equation (3.46) which can then be linearised into a zero eigenvalue Schrödinger equation whose solutions are discussed. A Poisson equation constraint is derived by taking the divergence of the momentum equation to give

$$\Delta p = -u_{i,j}u_{j,i},$$

(3.99)
and this constraint equation is re-written in terms of the pressure Hessian $P$, the strain matrix $S$ and the vorticity magnitude $|\omega|$. A quaternionic form of the momentum equation is derived and earlier derived results such as the vorticity equation and constraint equation are verified using this method. Furthermore, Ohkitani result for the evolution of the vorticity stretching is re-written in a 4-vector framework as

\[
\frac{D^2 \mathbf{w}}{Dt^2} = -q_p \otimes \mathbf{w},
\]

(3.100)

and a Lagrangian advection equation for the pressure variable $q_p$ is derived

\[
\frac{Dq_p}{Dt} = q \otimes q_p + q_{\mu,\lambda,\epsilon},
\]

(3.101)

although three new scalars have had to be introduced which are determined by the constraint equation (3.99). It is equations (3.98), (3.100), (3.101) and the quaternionic form of the evolution of the vorticity (2.32) that govern the vorticity dynamics at all points and times providing, of course, that their solutions remain finite.

The relative advantages and disadvantages of how to define the $\alpha$ and $\chi$ variables are highlighted when two different approaches are considered for the flow of a viscous fluid governed by the Navier-Stokes equations.
Chapter 4

The Euler equations with rotation

The momentum equation stated in equation (3.1), which was the starting point of the research that took place in the previous chapter, holds only in an inertial reference frame, and hence is not practical for modelling or deriving equations of motion that are valid on a rotating body (e.g. the Earth). Therefore, a non-inertial, or rotating, reference frame that rotates at a constant angular velocity must be considered. For these applications we also include a simple model of gravitational effects. The Euler equations will, therefore, be re-derived with the added effects of rotation (and gravity) being taken into account. Once this is achieved the remaining part of this chapter will attempt to place this particular dynamical system in the general framework of Chapter 2.

4.1 Equations of motion

The first change that is made to the momentum equation for an inviscid fluid is the addition of an external potential $\phi(x)$, typically representing the effects of gravity, so that the momentum equation (3.1) is given by

$$\frac{D_I u_I}{Dt} = -\frac{1}{\rho} \nabla p - \nabla \phi,$$

where $D_I/Dt$ and $u_I$ are the material derivative operator and the velocity field respectively in an inertial reference frame. We transform this equation to one which
is valid in a non-inertial frame, which has the same origin and rotates at a constant angular velocity $\Omega$. This is achieved by considering the following relationship between variables in the inertial reference frame and non-inertial reference frame

$$\frac{D_I}{Dt} = \frac{D_R}{Dt} + \Omega \times, \quad u_I = u_R + \Omega \times r,$$

(4.2)

where $D_R/Dt$ and $u_R$ are the material derivative operator and the velocity field in the rotating reference frame respectively and $r$ is the position vector from the centre of the Earth. Substituting equation (4.2) into (4.1) gives

$$\frac{D_R}{Dt} \Omega \times r + \frac{D_R u_R}{Dt} + \Omega \times \frac{D_R r}{Dt} + \Omega \times u_R + \Omega \times (\Omega \times r) = -\frac{1}{\rho} \nabla p - \nabla \phi,$$

(4.3)

The first term in (4.3) is zero under the assumption that the angular velocity is constant. Noting that $u_R = D_R r/Dt$, then the momentum equation for a fluid in a reference frame attached to the rotating Earth, where all subscripts have been dropped, is

$$\frac{Du}{Dt} + \Omega \times (\Omega \times r) + 2\Omega \times u = -\frac{1}{\rho} \nabla p - \nabla \phi;$$

(4.4)

the additional terms on the left-hand side of (4.4) are the Coriolis ($2\Omega \times u$) and centrifugal ($\Omega \times (\Omega \times r)$) forces. The next step is to combine the centrifugal force with the Newtonian gravity ($-\nabla \phi$) to give the apparent gravity term ($-\nabla \Phi$). This is achieved by expressing the centrifugal acceleration term as the gradient $\nabla \phi_c$ where the potential $\phi_c = \frac{-1}{2} \tilde{r}^2 \Omega^2$ and $\tilde{r}$ is the perpendicular distance to the Earth’s rotation axis. The momentum equation (4.4) then simplifies to

$$\frac{Du}{Dt} + 2\Omega \times u = -\frac{1}{\rho} \nabla p - \nabla \Phi.$$

(4.5)
4.2 The vorticity equation

To derive the corresponding expression for the vorticity associated with the momentum equation (4.5), the following forms of the equations are considered

\[
\frac{\partial u}{\partial t} + (u \cdot \nabla) u + 2\Omega \times u = -\frac{1}{\rho}\nabla p - \nabla \Phi, \tag{4.6}
\]

\[
\frac{\partial \rho}{\partial t} + (u \cdot \nabla) \rho + \rho (\nabla \cdot u) = 0, \tag{4.7}
\]

\[
\rho = \rho (p, \eta), \tag{4.8}
\]

\[
\frac{\partial \eta}{\partial t} + (u \cdot \nabla) \eta = 0. \tag{4.9}
\]

Expression (4.6) is the momentum equation (4.5) with the material derivative expanded into its local time derivative and non-linear advection term. The next equation, (4.7), is the statement of mass conservation, first mentioned in equation (3.2). At this point nothing is assumed about the flow under consideration and this is reiterated in the next statement (4.8) which says that the density is a function of both the pressure and the specific entropy \( \eta \). Finally, (4.9) is the corresponding prognostic equation for the specific entropy.

In a similar way in which the the non-linear advection term in the momentum equation was re-written in the previous section for the inertial, Euler equations (3.5), equation (4.6) can be restated as

\[
\frac{\partial u}{\partial t} + (\omega + 2\Omega) \times u + \frac{1}{2} \nabla (|u|^2) = -\frac{p}{\rho^2} \nabla \rho + \nabla \left( \frac{p}{\rho} \right) - \nabla \Phi, \tag{4.10}
\]

where the result \( \nabla (p/\rho) = (1/\rho) \nabla p - (p/\rho^2) \nabla \rho \) has been used to re-write the pressure gradient term. Introduce the absolute vorticity \( \xi \) in the rotational frame as

\[
\xi = \nabla \times u + 2\Omega = \left( \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}, \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} + 2\Omega_h, \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} + 2\Omega_v \right), \tag{4.11}
\]

where \( \Omega_h \) and \( \Omega_v \) are the horizontal and vertical components of the Earth’s rotation vector near \( x = 0 \). Re-writing equation (4.10) using the absolute vorticity and combining the gradient terms gives
\[
\frac{\partial \mathbf{u}}{\partial t} + \mathbf{\xi} \times \mathbf{u} = -\nabla P' - \frac{p}{\rho^2} \nabla \rho, \tag{4.12}
\]

where \(P'\) is the potential

\[
P' = \Phi + \frac{p}{\rho} + \frac{1}{2} |\mathbf{u}|^2. \tag{4.13}
\]

The corresponding vorticity equation is now derived by taking the curl of (4.12), which gives

\[
\frac{\partial \mathbf{\xi}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{\xi} - (\mathbf{\xi} \cdot \nabla) \mathbf{u} + \mathbf{\xi} (\nabla \cdot \mathbf{u}) = -\nabla \times \left( \frac{p}{\rho^2} \nabla \rho \right). \tag{4.14}
\]

Note that the divergence term \((\nabla \cdot \mathbf{u})\) is no longer zero and is in fact given by the mass conservation equation (4.7), which upon re-arranging gives

\[
\nabla \cdot \mathbf{u} = -\frac{1}{\rho} \left( \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla \right) \rho,
\]

substituting into equation (4.14) gives

\[
\left( \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla \right) \mathbf{\xi} - \frac{\mathbf{\xi}}{\rho} \left( \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla \right) \rho - (\mathbf{\xi} \cdot \nabla) \mathbf{u} = -\nabla \times \left( \frac{p}{\rho^2} \nabla \rho \right). \tag{4.15}
\]

Furthermore, note the following two results needed to simplify the above expression

\[
\rho \frac{D}{Dt} \left( \frac{\mathbf{\xi}}{\rho} \right) = \frac{D \mathbf{\xi}}{Dt} - \frac{\mathbf{\xi} D \rho}{\rho D t}, \quad \nabla \times \left( \frac{p}{\rho^2} \nabla \rho \right) = -\frac{1}{\rho^2} \nabla \rho \times \nabla p.
\]

Substituting these result into (4.15), the general vorticity equation for a fluid governed by equations (4.6)-(4.9) is given by

\[
\frac{D}{Dt} \left( \frac{\mathbf{\xi}}{\rho} \right) = \left[ \left( \frac{\mathbf{\xi}}{\rho} \right) \cdot \nabla \right] \mathbf{u} + \frac{1}{\rho^2} \nabla \rho \times \nabla p. \tag{4.16}
\]

The first term on the right-hand side of (4.16) represents the stretching and tilting of the quantity \((\mathbf{\xi}/\rho)\). The second term is the baroclinic term and represents the pressure-torque of the fluid flow. With the vorticity equation in this form it is now possible to consider a range of different flows, the first is the special case of a constant density fluid.
Chapter 4. The Euler equations with rotation

4.3 Constant density fluid

For the particular case of a constant uniform density flow, equation (4.16) simplifies to

\[ \frac{D\xi}{Dt} = (\xi \cdot \nabla) u. \]  

(4.17)

This is because the baroclinic term \((\nabla \rho \times \nabla p)\) is identically zero as the density is a constant. Furthermore, the representation of this density in the remaining two terms can be incorporated into the pressure gradient in an identical way as was seen in the previous section. Therefore, the ratio \((\xi / \rho)\) in the vorticity equation (4.16) is simply the absolute vorticity \(\xi\). For the case of a constant density fluid in a rotating frame the statement of mass conservation simplifies to the incompressibility constraint (3.3).

For this particular problem, the vorticity stretching vector \(\sigma\) is given by \((\xi \cdot \nabla) u\). However, because the problem is set in a rotating reference frame it is no longer possible to express the vorticity stretching vector in terms of the strain matrix because the vorticity is no longer strictly the curl of the velocity field. The stretching rate and alignment vector are now given by

\[ \alpha = \frac{\xi \cdot (\xi \cdot \nabla) u}{\xi \cdot \xi}, \quad \chi = \frac{\xi \times (\xi \cdot \nabla) u}{\xi \cdot \xi}. \]  

(4.18)

Although it is no longer possible to incorporate the strain matrix into the expressions for \(\alpha\) and \(\chi\), one important result still holds from the analysis of the previous chapter on the incompressible non-rotating Euler equations. As the evolution equation for the absolute vorticity is given by (3.8), albeit with \(\omega\) replaced by \(\xi\), the Ertel result (3.27) holds. Therefore to calculate the derivative of the vortex stretching and to close the problem for the 4-vector \(q\) an expression comprising the three components of the material derivative \(Du/Dt\) for the rotational Euler equations needs to be derived. The momentum equation for the constant density, Euler equations with rotation is given by
\[
\frac{Du}{Dt} = -2\Omega \times u - \nabla p - \nabla \Phi,
\]  
(4.19)

Substituting equation (4.19) into (3.27) gives

\[
\frac{D\sigma}{Dt} = (\xi \cdot \nabla) \frac{Du}{Dt} = (\xi \cdot \nabla) \left[ -2\Omega \times u - \nabla p - \nabla \Phi \right],
\]

\[
= -2\Omega \times (\xi \cdot \nabla) u - P\xi - \tilde{\Phi} \xi.
\]  
(4.20)

Combining the pressure Hessian matrix \( P \) with the external potential matrix \( \tilde{\Phi} \) into a single Hessian matrix \( P' \) and substituting the expression for the vortex stretching vector into (4.20) gives the result that the Lagrangian derivative of the vortex stretching is given by

\[
\frac{D\sigma}{Dt} = -P'\xi - 2\Omega \times \sigma.
\]  
(4.21)

Note that the effect of adding rotation to the problem is the additional vector product term of the angular velocity acting on the vortex stretching vector. Define similar variables to those in (3.35) but in terms of the new "modified" pressure Hessian matrix \( P' \)

\[
\alpha_{p'} = \frac{\xi \cdot P'\xi}{\xi \cdot \xi}, \quad \chi_{p'} = \frac{\xi \times P'\xi}{\xi \cdot \xi}.
\]  
(4.22)

The evolution equations for the \( \alpha \) and \( \chi \) variables in a rotating frame are obtained by substituting (4.20) into (2.18) and (2.20) to give

\[
\frac{D\alpha}{Dt} = \chi^2 - \alpha^2 - \alpha_{p'} - \frac{\xi \cdot (2\Omega \times \sigma)}{\xi \cdot \xi},
\]  
(4.23)

\[
\frac{D\chi}{Dt} = -2\chi\alpha - \chi_{p'} - \frac{\xi \times (2\Omega \times \sigma)}{\xi \cdot \xi}.
\]  
(4.24)
4.3.1 The quaternionic formulations of the equation dependent term

These two equations (4.23) and (4.24), derived in the previous section, for the stretching and spin rates differ from the form they took in the inertial reference frame, simply because, for this problem, rotation has been added and more will be said about these equations later. Regardless of the specific form of these equations of motion, they can be combined into a single 4-vector $\mathbf{q}$ and its evolution is given in equation (2.29) and is stated below

$$\frac{D\mathbf{q}}{Dt} + \mathbf{q} \otimes \mathbf{q} + s_1 \otimes s_2 = 0,$$

where now $s_1$, the first variable in the equation dependent term, is given explicitly for this problem by

$$s_1 = (0, -P'\xi - 2\Omega \times \sigma)^T,$$  \hspace{1cm} (4.25)

the other two terms $\mathbf{q}$ and $s_2$ are defined by the terms in equation (2.28).

Recall that the equation dependent term $s_1 \otimes s_2$ provides information regarding certain dependent variables attributed to the flow, for example in this problem the pressure, the absolute vorticity and the external potential. This term also explicitly incorporates the effects of adding rotation to the problem. Note that the flow dependent term can be easily decomposed into different components each highlighting the different terms within the flow. For example,

$$s_1 \otimes s_2 = q_{p'} - \tau \otimes s_2,$$  \hspace{1cm} (4.26)

where $q_{p'} = (\alpha_{p'}, \chi_{p'})^T$ and $\tau = (0, 2\Omega \times \sigma)^T$. The following representation of the 4-vector $\mathbf{q}$ in terms of a pseudo-angular velocity 4-vector highlights one of the advantages of formulating these equations using quaternions - their versatility i.e. there is more than one way of representing the same quaternion. To further exploit this fact, consider equations (4.23) and (4.24) re-written using a combination of vector identities and basic results from matrix algebra (see glossary), to give
\[
\frac{D \alpha}{Dt} + \alpha^2 - |\chi|^2 + \alpha \chi - 2\Omega \cdot \chi = 0, \quad (4.27)
\]
\[
\frac{D \chi}{Dt} + 2\chi + \chi \chi' + 2\alpha \Omega + 2 \times \chi = \frac{(\sigma \cdot 2\Omega) \xi}{\xi \cdot \xi}. \quad (4.28)
\]

Once again combining these two equations into the single equation in terms of the vector \( q \) gives

\[
\frac{D q}{Dt} + q \otimes q + q \Omega \otimes q = \tilde{q} \chi', \quad (4.29)
\]

where the angular velocity 4-vector \( \Omega = (0, 2\Omega)^T \), and \( \tilde{q} \chi' \) is given by

\[
\tilde{q} \chi' = \left( -\alpha \chi', \frac{(\sigma \cdot 2\Omega) \xi}{\xi \cdot \xi} - \chi \chi' \right)^T.
\]

If the 4-vector evolution equation for the inertial incompressible Euler equations (3.36) is compared to the rotational form (4.29), the obvious differences are, first, the new product term \( q \otimes \Omega \otimes q \), which is due to adding rotation to the original momentum equations. Secondly, the 4-vector \( \tilde{q} \chi' \) does not represent the pressure and external potential terms alone but also incorporates an additional term, due to rotation. This term is a product of the subtle geometry of the double cross product that manifests itself in the evolution equation for the vortex alignment vector (4.28).

### 4.3.2 The Ohkitani result in 4-vector form

The previous section has highlighted the many different ways that the equation dependent term in the equation for \( q \) can be expressed mathematically. This is due to the versatile nature in which it is possible to represent a particular quaternion and the logic behind expressing them in a number of different ways is to see the role that each particular dynamical variable plays in the governing equations. One major disadvantage of these formulations is that the expression for the evolution of the vortex stretching - a variant of the Ohkitani relation - is written strictly in terms of 3-vectors when the variables in the \( q \)-equation are in 4-vectors. In this section the modified Ohkitani result (4.21) is re-written solely in 4-vector form. Two major
advantages of this are, first, to bring the problem within a general framework of
4-vector algebra and secondly, the evolution equations for \( q \) can be expressed in a
more concise way in terms of the quaternionic product operator \( \otimes \) without any of
the complicated algebra seen in equations (4.25) and (4.29).

Recall two key result from Chapter 2, which are that the vorticity 4-vector \( w \)
and the 4-vector \( q \) evolve according to

\[
\frac{Dw}{Dt} = q \otimes w, \quad \frac{Dq}{Dt} + q \otimes q + \frac{1}{w \cdot w} \frac{D^2w}{Dt^2} \otimes w = 0. \tag{4.30}
\]

The Ohkitani result for this particular problem can then be re-written as

\[
\frac{D^2\xi}{Dt^2} = -P'\xi - 2\Omega \times (\xi \cdot \nabla) u. \tag{4.31}
\]

The first term on the right-hand side of equation (4.31) can be expressed as
\(-P'\xi = -q_{p'} \otimes w\) where \( w \) is now the absolute vorticity 4-vector and the second term is

\[
2\Omega \times (\xi \cdot \nabla) u = \frac{1}{2} \left( q_{\Omega} \otimes \frac{Dw}{Dt} - \frac{Dw}{Dt} \otimes q_{\Omega} \right). \tag{4.32}
\]

Substituting in the expression for the evolution of \( w \) given in equation (4.30) then
the Ohkitani result in 4-vector form is given by

\[
\frac{D^2w}{Dt^2} = -q_{p'} \otimes w - \frac{1}{2} \left( q_{\Omega} \otimes (q \otimes w) - (q \otimes w) \otimes q_{\Omega} \right). \tag{4.33}
\]

This result when substituted into the second equation in (4.30) gives the follow-
ing evolution equation for the 4-vector \( q \) defined for the constant density, Euler
equations with rotation

\[
\frac{Dq}{Dt} + q \otimes q + \frac{1}{w \cdot w} \left\{ -q_{p'} \otimes w - \frac{1}{2} \left( q_{\Omega} \otimes (q \otimes w) - (q \otimes w) \otimes q_{\Omega} \right) \right\} \otimes w = 0. \tag{4.34}
\]

This equation is consistent with the two previous expressions for the \( q \)-vector but
in the above form this is strictly in terms of the four 4-vectors \( q, q_{p'}, w \) and \( q_{\Omega} \)
which are the vorticity stretching rate and spin variables, the pressure variables, the
absolute vorticity and the rotation vector respectively. In any of these formulations a constraint equation must be derived to provide a link between these dependent variables as they are not all independent of each other. Before moving onto deriving this equation a brief mention is made of the corresponding complex structure for this particular problem.

### 4.3.3 Brief mention of the corresponding complex structure

Although the corresponding complex structure and their linearisation to a set of equations that are solvable is not the main driving factor in this particular research problem a brief mention will be made of how this particular system could be adapted for that context and in fact takes a very similar form to the case of the incompressible inertial Euler equations of the previous chapter.

The evolution of $|\chi|$ is given by taking the scalar product of (4.28) with the vortex alignment vector $\chi$ to give the result that

$$
\frac{D|\chi|}{Dt} + 2|\chi|\alpha + 2\alpha \Omega \cdot \hat{\chi} = \left[ \frac{(\sigma \cdot 2\Omega)}{\xi \cdot \xi} - \chi_p' \right] \cdot \hat{\chi},
$$

(4.35)

and the remaining analysis is identical to that seen in the previous chapter. The key differences, in the corresponding results, are that the complex variables are now in terms of the components of the modified pressure variable $q_p'$ and in the final part of the analysis, when the zero-eigenvalue Schrödinger equation is derived, the rotational vector $(2\Omega)$ appears within the potential term, which of course, plays the pivotal role in the solution to zero-eigenvalue problem.

### 4.3.4 The corresponding constraint equation

The constraint equation, vital due to the increase in the number of prognostic vorticity equations from three to four which define the dynamical system, also give a unique relationship between the dependent variables seen in the evolution equations for the vortex stretching rate and the vortex alignment vector. The corresponding
Chapter 4. The Euler equations with rotation

65

Constraint equation for this problem is once again derived by considering the divergence of the momentum equation (4.19), which is given by

$$\nabla \cdot \left( \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) = -\nabla \cdot (2\mathbf{\Omega} \times \mathbf{u}) - \nabla \cdot (\nabla p + \nabla \Phi),$$

(4.36)

using the results derived in the previous chapter concerning the constraint equation for the incompressible inertial Euler equations, (4.36) simplifies to

$$u_{j,i}u_{i,j} - 2\Omega \cdot \text{curl} \mathbf{u} = -\Delta p - \Delta \Phi.$$  

(4.37)

This Poisson equation - known as the balance equation - is the specific mathematical relationship between the dependent variables of the $\alpha$ and $\chi$ terms, in this case, the pressure ($-\Delta p$), the apparent gravity ($-\Delta \Phi$), the angular velocity ($2\Omega$) and the vorticity ($u_{j,i}u_{i,j}$). It is possible to conclude that the 4-vector expressions $\mathbf{q}$, $\mathbf{q}^\prime$ and $\mathbf{q}_\Omega$ in equation (4.34) are not independent of one another.

4.3.5 Beale-Kato-Majda calculation for the Euler equations with rotation

The magnitude of the vorticity $|\xi|$ for the incompressible Euler equations with rotation is given by

$$\frac{\partial |\xi|}{\partial t} + \mathbf{u} \cdot \nabla |\xi| = \alpha |\xi|,$$

(4.38)

therefore the equivalent expression to equation (3.24), regarding the constraint on the development of singular solutions to the incompressible, rotational Euler equations is

$$||\xi(\cdot, t)||_m \leq |\xi(0)| \exp \int_0^t ||\alpha(\cdot, \tau)||_\infty d\tau.$$  

(4.39)

In the previous section a number of different ways of expressing the evolution equations and quaternionic formulation for the $\alpha$ and $\chi$ variables was discussed, each having their own advantages and disadvantages. However, a further consideration
of these equations is needed. This is so certain explicit calculations regarding the criteria for singular solutions, which were developed in the earlier analysis of the inertial, Euler equations, can be further exploited, this time for the rotational problem. Recall, the evolution equations for the stretching rate and the vortex alignment vector given by

\[
\frac{D\alpha}{Dt} = |\chi|^2 - \alpha^2 - \frac{\xi \cdot P' \xi}{\xi \cdot \xi} - \frac{\xi \cdot (2\Omega \times \sigma)}{\xi \cdot \xi},
\]

(4.40)

\[
\frac{D\chi}{Dt} = -2\chi \alpha - \frac{\xi \times P' \xi}{\xi \cdot \xi} - \frac{\xi \times (2\Omega \times \sigma)}{\xi \cdot \xi},
\]

(4.41)

The numerators in both the pressure Hessian and rotational term in equations (4.40) and (4.41) can be combined as

\[
P' \xi + 2\Omega \times (\xi \cdot \nabla) \mathbf{u} = P' \xi + 2\Omega \times (\nabla \mathbf{u}) \xi
\]

\[
= P' \xi + \left( \left( 2\Omega \times (\nabla \mathbf{u}) \right) \right)_{ij} \xi, 
\]

(4.42)

where \(\nabla \mathbf{u}\) is the velocity gradient matrix \(u_{i,j}\), \((\nabla \mathbf{u})_j\) is the \(j^{\text{th}}\)-column of the matrix \((\nabla \mathbf{u})\) and \(\left( 2\Omega \times (\nabla \mathbf{u}) \right)_i\) is the \(i^{\text{th}}\)-component of the vector product \(2\Omega \times (\nabla \mathbf{u})_j\).

The rotational term above can be re-written as \(\Omega^* \xi\) where

\[
\Omega^* = \begin{pmatrix}
2\Omega_h \frac{\partial w}{\partial x} - 2\Omega_v \frac{\partial w}{\partial y} & 2\Omega_h \frac{\partial w}{\partial y} - 2\Omega_v \frac{\partial w}{\partial x} & 2\Omega_h \frac{\partial w}{\partial z} - 2\Omega_v \frac{\partial w}{\partial x} \\
2\Omega_v \frac{\partial u}{\partial x} & 2\Omega_v \frac{\partial u}{\partial y} & 2\Omega_v \frac{\partial u}{\partial z} \\
-2\Omega_h \frac{\partial u}{\partial x} & -2\Omega_h \frac{\partial u}{\partial y} & -2\Omega_h \frac{\partial u}{\partial z}
\end{pmatrix}.
\]

(4.43)

The evolution of the vorticity stretching vector is then simply

\[
\frac{D}{Dt} (\xi \cdot \nabla) \mathbf{u} = -\bar{P} \xi \quad \text{where} \quad \bar{P} = P + \Omega^*.
\]

(4.44)

The main advantage of writing the evolution of the vortex stretching in terms of a single square matrix is for the simplicity and ease at which it will be possible to verify numerically the results that are being derived in this section. However, by
incorporating all terms into a single matrix it is not really that clear what part certain individual terms play in respect of the theoretical analysis and our understanding.

Defining new variables \( \alpha_p \) and \( \chi_p \)

\[
\alpha_p = \frac{\xi \cdot \bar{P}\xi}{\xi \cdot \xi}, \quad \chi_p = \frac{\xi \times \bar{P}\xi}{\xi \cdot \xi},
\]  
\[\text{(4.45)}\]

and then the following result which was seen (albeit in terms of these pressure-rotation variables) in the previous chapter in (3.63) becomes

\[
\frac{D}{Dt} \left[ \frac{1}{2} (\alpha^2 + |\chi|^2) \right] = -\alpha (\alpha^2 + |\chi|^2) - \alpha\alpha_p - \chi \cdot \chi_p.
\]  
\[\text{(4.46)}\]

For a positive vortex stretching rate the following inequality holds

\[
|| \alpha (\cdot, t) ||_{\infty} \leq || \chi (\cdot, t) ||_{\infty} \leq 2 \int_0^t || \chi (\cdot, \tau) ||_{\infty} d\tau.
\]  
\[\text{(4.48)}\]

These results are analogous to those in (3.65) and (3.66). Hence the \( L^\infty \)-norm of the vorticity is no longer bounded by strictly the maximum eigenvalue of the pressure Hessian but also by the additional rotational matrix \( \Omega^* \).

### 4.4 Analysis for a barotropic fluid

For a barotropic fluid the density is a function of the pressure, \( \rho = \rho (p) \), hence in the general vorticity equation (4.16) the baroclinic term \( (\nabla \rho \times \nabla p) \) is zero and (4.16) simplifies to

\[
\frac{D}{Dt} \left( \frac{\xi}{\rho} \right) = \left\{ \left( \frac{\xi}{\rho} \right) \cdot \nabla \right\} u,
\]  
\[\text{(4.49)}\]

re-writing the ratio \( (\xi/\rho) \) as \( w \) gives the familiar form for the vorticity equation, seen in (4.17), as
\[ \frac{D\mathbf{u}}{Dt} = (\mathbf{w} \cdot \nabla) \mathbf{u}. \]  

(4.50)

For the case of a barotropic fluid, the flow can be both compressible and incompressible. The analysis is going to be restricted to the incompressible case but could easily be modified to take into account the flow of a compressible fluid.

### 4.4.1 Incompressible case

For an incompressible flow, \( \text{div} \mathbf{u} = 0 \). Applying this to the statement of mass conservation (4.7), implies

\[ \left( \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla \right) \rho = 0, \]

(4.51)

therefore the density is a conserved, but not constant, quantity of the flow. Also, as the flow is barotropic, \( \frac{\partial \rho}{\partial p} \neq 0 \), equation (4.51) yields the result that

\[ \frac{Dp}{Dt} = 0, \]

and the pressure is a further conserved quantity of the flow. The momentum equation for an incompressible, barotropic flow is

\[ \frac{Du}{Dt} = -2\Omega \times \mathbf{u} - \frac{1}{\rho} \nabla p - \nabla \Phi, \]

(4.52)

where \( \rho = \rho(p) \). The corresponding vorticity equation can in fact be further simplified to the one given in equation (4.50). This is achieved by substituting the incompressible constraint into equation (4.14) and, in fact, the vorticity equation for an incompressible, barotropic fluid is given by equation (4.17), which is the corresponding vorticity equation for a constant density flow. The equation derived in (4.50) can be seen as the corresponding vorticity equation for a compressible, barotropic flow. The analysis of the vortex stretching rate and the vortex alignment vector is therefore identical to that of the preceding section concerning a constant density fluid. The stretching rate and alignment vector are given by (4.18) and the Ertel result, which enables the calculation of the evolution equations for \( \alpha \) and \( \chi \),
Chapter 4. The Euler equations with rotation

once again holds. However the evolution equation for the vorticity stretching vector given in (4.21) no longer holds and must be re-derived. In fact, it is the form of the momentum equation (and specifically the fact that the density can not easily be incorporated into the pressure gradient term) that changes the subsequent analysis of this problem and not the form of the vorticity equation. Substituting (4.52) into (3.27) gives

$$\frac{D\sigma}{Dt} = (\mathbf{\xi} \cdot \nabla) \left[ -2\Omega \times \mathbf{u} - \rho(p)^{-1} \nabla p - \nabla \Phi \right],$$

$$= -2\Omega \times (\mathbf{\xi} \cdot \nabla) \mathbf{u} - \left( \frac{1}{\rho} P + \tilde{\Phi} \right) \mathbf{\xi} + \frac{1}{\rho^2} \nabla p (\mathbf{\xi} \cdot \nabla) \rho,$$

$$= -2\Omega \times (\mathbf{\xi} \cdot \nabla) \mathbf{u} - P\mathbf{\xi} + \frac{1}{\rho^2} \nabla p (\mathbf{\xi} \cdot \nabla) \rho,$$

(4.53)

where the Hessian matrix \(\tilde{P} = \frac{1}{\rho} P + \tilde{\Phi}\). With respect to a constant density flow, the equation of motion of the vorticity stretching vector \(\sigma = (\mathbf{\xi} \cdot \nabla) \mathbf{u}\) for a barotropic fluid has an additional term, and is in fact the product of the advected density (driven by the vorticity) with the pressure gradient. From the earlier analysis of the constant density case is, can the evolution of the vortex stretching be formulated in a 4-vector framework? The Hessian matrix terms can be expressed using the result that for a matrix \(A\), 3-vector \(\mathbf{b}\) and 4-vectors \(\mathbf{a}, \mathbf{b}\) then

$$\mathbf{b} \cdot \mathbf{a} \times \mathbf{b} = \mathbf{a} \otimes \mathbf{b},$$

where \(\mathbf{a} = \left( \mathbf{b} \cdot \mathbf{b}, \mathbf{b} \times \mathbf{b} \right)\),

(4.54)

where \(\mathbf{b}\) is the 4-vector representation of the 3-vector \(\mathbf{b}\). Although this result has not been explicitly mentioned before it has been implied almost from the calculations of the preceding sections and chapters. The additional barotropic term can be expressed in quaternionic form by introducing the density 4-vector \(\mathbf{p} = (\rho, 0)^T\) and then

$$\frac{1}{\rho^2} \nabla p (\mathbf{\xi} \cdot \nabla) \rho = \left\{ p^{-2} \otimes (\nabla \otimes \mathbf{p}) \right\} \otimes \left\{ (\mathbf{u} \cdot \nabla) \otimes \mathbf{p} \right\},$$

(4.55)

where \(p^{-2} = \rho^{-2} (1, 0)^T\). Combining all these expressions together then in quaternionic form the evolution of the vortex stretching for an incompressible, barotropic
flow is given by

$$\frac{D^2 \mathbf{w}}{Dt^2} = -\frac{1}{2} \left( \mathbf{q} \otimes (\mathbf{q} \otimes \mathbf{w}) - (\mathbf{q} \otimes \mathbf{w}) \otimes \mathbf{q} \right) - \mathbf{q}_{\beta} \otimes \mathbf{w} + \left\{ p^{-2} \otimes (\nabla \otimes \mathbf{p}) \right\} \otimes \left\{ (\mathbf{w} \cdot \nabla) \otimes p \right\}, \quad (4.56)$$

where $\mathbf{q}_{\beta}$ is the single 4-vector representation of the $(\alpha, \chi)$ variables in terms of the Hessian matrix $\hat{P}$. Substituting this second result into equation (4.30) will once again given the full form of the evolution of the 4-vector $\mathbf{q}$ for this case of an incompressible barotropic flow.

### 4.4.2 Constraint equation for a barotropic flow

The constraint equation for an incompressible, barotropic flow is given by

$$u_{j,i}u_{i,j} = -2\Omega \cdot \text{curl} \mathbf{u} - \frac{1}{\rho} \Delta p + \frac{1}{\rho^2} |\nabla p|^2 - \Delta \Phi, \quad (4.57)$$

### 4.5 Summary

This chapter has considered the general quaternionic formulation of the vorticity variables applied to the Euler equations with rotation. The vorticity equation was derived in such a way that a number of different flows could be considered. This chapter dealt specifically with the two cases of a constant density and incompressible, barotropic flow.

In both cases, Ertel’s theorem could be applied to define a specific form for the evolution of the vortex stretching and for the case of a constant density flow a number of different variations of the 4-vector equations for $\mathbf{q}$ were derived. The justification of expressing these equations in a number of ways was two-fold. The first was to highlight the versatility of a quaternionic formulation and the second was to express the relative effects that each dependent variable plays within the particular dynamical system. However, two of these expression were of a greater
importance than the others. The first transpired by writing the modified Ohkitani result solely in terms of 4-vectors and so moving away from a 3-vector representation (of $D^2 \mathbf{w}/Dt^2$), in a completely 4-vector ($D \mathbf{q}/Dt$) system. This formulation eliminated the need for the quite complicated algebra manipulation of certain 3-vectors and is in fact a more mathematically correct way of expressing the dynamical system in terms of only the four 4-vectors $\mathbf{q}, \mathbf{q}', \mathbf{q}_\Omega, \mathbf{w}$.

The second formulation expressed the evolution of the vortex stretching in terms of a single Hessian matrix $\tilde{P}$. The obvious advantages of this will be seen later when the results derived in this chapter for the potential development of singular solutions will be considered numerically. Furthermore, the differences in the two systems were noted. Surprisingly, they did not manifest in the vorticity equation but in the momentum equations and hence the vortex stretching and corresponding constraint equations.

This chapter has really highlighted the obvious advantage of writing the evolution of the vortex stretching vector in 4-vector form so as to be consistent with the governing equations for the 4-vector $\mathbf{q}$ and to not introduce any unwanted 3-vector based terms. As a result of this chapter it would be to insist in a retrospective way that, where possible, that the evolution of the vortex stretching is expressed solely in terms of previous defined 4-vectors that play a significant role in the dynamical system.

In this chapter we have failed to consider the case of a baroclinic flow but instead of tackling this particular problem now, the focus of the research shifts to trying to apply the constraint of hydrostatic balance to the two cases already discussed in this chapter (the baroclinic problem is, in fact, considered in Chapter 6).
Chapter 5

The breakdown of the hydrostatic case

In this chapter, the particular constraint of adding hydrostatic balance to the problem is considered. The result of this is interesting in two ways. First, the form of the vorticity equation does not take the "standard" vortex stretching form seen previously, and secondly the quaternionic/4-vector structure breaks down.

5.1 Momentum and vorticity equations

Consider the momentum equation (4.6) in component form

\[ \frac{\partial u}{\partial t} + (u \cdot \nabla) u - 2\Omega v + 2\Omega w = -\frac{1}{\rho} \frac{\partial p}{\partial x} - \frac{\partial \Phi}{\partial x}, \]  \tag{5.1} \\
\[ \frac{\partial v}{\partial t} + (u \cdot \nabla) v + 2\Omega u = -\frac{1}{\rho} \frac{\partial p}{\partial y} - \frac{\partial \Phi}{\partial y}, \]  \tag{5.2} \\
\[ \frac{\partial w}{\partial t} + (u \cdot \nabla) w - 2\Omega h = -\frac{1}{\rho} \frac{\partial p}{\partial z} - \frac{\partial \Phi}{\partial z}. \]  \tag{5.3} 

In the traditional approximation to this set of equations, the horizontal component of the Earth’s rotation vector is neglected, and it is worth noting that this should be justified for each particular case. Both horizontal terms in (5.1) and (5.3) must be cancelled or the resulting equations fail to conserve energy. The remaining angular
Chapter 5. The breakdown of the hydrostatic case

velocity term $2\Omega v$ is denoted by $f$, and is called the Coriolis parameter (and here will be taken to be a constant). Finally, the surface of the Earth is approximately a geopotential surface, that is, a surface of constant $\Phi$. Therefore, the geopotential gradient $\nabla \Phi$ is in effect the gravitational acceleration $g$, and is normal to the surface of the Earth. Mathematically this is $\nabla \Phi = (0, 0, g)$. Equations (5.1)-(5.3) then become

\[
\begin{align*}
\frac{\partial u}{\partial t} + (u \cdot \nabla) u - fv & = -\frac{1}{\rho} \frac{\partial p}{\partial x}, \\
\frac{\partial v}{\partial t} + (u \cdot \nabla) v + fu & = -\frac{1}{\rho} \frac{\partial p}{\partial y}, \\
\frac{\partial w}{\partial t} + (u \cdot \nabla) w & = -\frac{1}{\rho} \frac{\partial p}{\partial z} - g. 
\end{align*}
\] (5.4)

For a wide range of space and time values, the vertical component of the momentum equation (5.6) is dominated by the contribution of the pressure gradient force and the buoyancy force; the atmosphere is approximately in hydrostatic balance. The governing equation for the vertical component in this state of balance is

\[
0 = -\frac{\partial p}{\partial z} - \rho g.
\] (5.7)

To form the corresponding vorticity equation for the set of momentum equations (5.4), the non-linear advection terms (5.5) and (5.7) are re-written as

\[
\begin{align*}
\frac{\partial u}{\partial t} + \frac{1}{2} \frac{\partial}{\partial x} (u^2 + v^2) - v \left( f + \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) + w \frac{\partial u}{\partial z} + \frac{1}{\rho} \frac{\partial p}{\partial x} & = 0, \\
\frac{\partial v}{\partial t} + \frac{1}{2} \frac{\partial}{\partial y} (u^2 + v^2) + u \left( f + \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) + w \frac{\partial v}{\partial z} + \frac{1}{\rho} \frac{\partial p}{\partial y} & = 0; 
\end{align*}
\] (5.8)

(5.9)

for the vertical component (5.7), the gradient term seen in equations (5.8) and (5.9) needs to be incorporated into equation (5.7) to give

\[
\frac{1}{2} \frac{\partial}{\partial z} (u^2 + v^2) + \frac{1}{\rho} \frac{\partial p}{\partial z} = -g + u \frac{\partial u}{\partial z} + v \frac{\partial v}{\partial z}. 
\] (5.10)
The three terms in equations (5.8)-(5.10) can be written as a single 3-vector equation

\[
\frac{\partial \mathbf{v}}{\partial t} + \nabla \left( \frac{1}{2} u^2 + \frac{1}{2} v^2 + \frac{p}{\rho} \right) + k \times \zeta \mathbf{u} + w \frac{\partial u}{\partial z} + w \frac{\partial v}{\partial z} = -\frac{p}{\rho^2} \nabla p + \left( -g + u \frac{\partial u}{\partial z} + v \frac{\partial v}{\partial z} \right) \mathbf{k},
\]

where \( \mathbf{v} = (u, v) \) and \( \zeta \) is the vertical component of the absolute vorticity

\[
\zeta = f + \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}.
\]

Applying the three-dimensional curl operator to equation (5.11) and combining it with the statement of mass conservation (4.7) gives the corresponding vorticity equation

\[
\frac{D}{Dt} \left( \frac{\xi \rho}{\rho} \right) = \left[ \left( \frac{\xi \rho}{\rho} \right) \cdot \nabla \right] \mathbf{u} + \frac{1}{\rho^2} \nabla \rho \times \nabla p,
\]

where

\[
\xi = \left( -\frac{\partial v}{\partial z}, \frac{\partial u}{\partial z}, f + \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right).
\]

The problem of finding the evolution of the right-hand side of the vorticity equation (5.11) now has to be considered. This problem is in effect the same as for the problem of a viscous flow governed by the Navier-Stokes, mentioned in Chapter 3, or for the case of a baroclinic flow, derived from the Euler equations with rotation in Chapter 4, that is, the vorticity equation is no longer simply of the form given in equation (3.8). Consider a vorticity equation of the form

\[
\frac{D\omega}{Dt} = (\omega \cdot \nabla) \mathbf{u} + \epsilon (x, t),
\]

where \( \omega \) is a general three-dimensional vorticity and \( \epsilon \) is the non-vortex stretching part of the vorticity equation. Using the results derived in Chapter 3.4, the evolution of the vorticity stretching vector in (5.15) with the vector \( \mathbf{u} \) replaced by the scalar \( \mu \), is given by
\[
\frac{D}{Dt} (\omega_i \mu, i) = \epsilon_i \mu_i + \omega_k u_{i,k} \mu, i + \omega_i \frac{\partial}{\partial x_i} \left( \frac{D\mu}{Dt} \right) - \omega_i u_{j,i} \mu, j,
\]
\[
= \epsilon_i \mu, i + \omega_i \frac{\partial}{\partial x_i} \left( \frac{D\mu}{Dt} \right). \tag{5.16}
\]

The evolution of the right-hand side (\(\sigma\)) of equation (5.15) is then given by
\[
\frac{D^2 \omega}{Dt^2} = \frac{D\epsilon}{Dt} + (\epsilon \cdot \nabla) u + (\omega \cdot \nabla) \frac{Du}{Dt}. \tag{5.17}
\]
when \(\epsilon\) is specified, the exact value of its derivative can be calculated.

In the previous chapters, this non-stretching term was zero and it was possible to directly substitute expressions into (5.17) for the horizontal and vertical accelerations of the flow and then find the corresponding expressions for the evolution of the stretching rate and vortex alignment vector. This is because there existed an explicit expression for the material derivative in the three components of the velocity field \(u\). For the case of hydrostatic balance, regardless, at this point if the non-stretching term is zero or not, we have prognostic equations for the velocity in the two horizontal components but not the vertical one. It is therefore not possible, in the case of hydrostatic balance, to directly substitute and find an expression for the evolution of the right-hand side of the vorticity equation and hence equations for the evolution of \((\alpha, \chi)\). This problem, of not being able to directly apply Ertel’s theorem to obtain the second material derivative of the vorticity, will exist for any approximation that does not have explicit form for the acceleration in all three spatial components.

Instead of trying to build the approximations directly into the original momentum equations, and then attempting to derive the corresponding equations, a more logical route is to start with the original, un-approximated momentum equations, derive the evolution equations required and then make the approximations. One way of achieving this is to consider the non-dimensional form of the momentum and vorticity equations.
Chapter 5. The breakdown of the hydrostatic case

5.2 The non-dimensional momentum and continuity equations

This section considers (both the independent and dependent) variables and corresponding momentum and mass conservation equations in their non-dimensional form. Choose scales $L, D, T, P, U, W, \rho$ and $\tilde{g}$, which characterise the magnitudes of length, depth, time, pressure, horizontal and vertical velocities, density and gravity respectively. These scales are then used to define non-dimensional dependent and independent variables (denoted by primes), as follow:

\[
(x, y, z) = (Lx', Ly', Dz'),
\]

\[
t = Tt',
\]

\[
(u, v, w) = (Uu', Uv', Ww'),
\]

\[
p = Pp', \quad \rho = \rho \rho', \quad g = \tilde{g} \tilde{g}'.
\]

Substituting these expressions into the three separate momentum equations (5.4)-(5.6) gives

\[
\frac{U}{T} \frac{\partial u'}{\partial t'} + \frac{U^2}{L} \left\{ u' \frac{\partial u'}{\partial x'} + v' \frac{\partial u'}{\partial y'} \right\} + \frac{WU}{D} \left\{ w' \frac{\partial u'}{\partial z'} \right\} - fUv' = -\frac{1}{\rho \rho'} \frac{P}{L} \frac{\partial p'}{\partial x'}, \tag{5.19}
\]

\[
\frac{U}{T} \frac{\partial v'}{\partial t'} + \frac{U^2}{L} \left\{ u' \frac{\partial v'}{\partial x'} + v' \frac{\partial v'}{\partial y'} \right\} + \frac{WU}{D} \left\{ w' \frac{\partial v'}{\partial z'} \right\} + fUu' = -\frac{1}{\rho \rho'} \frac{P}{L} \frac{\partial p'}{\partial y'}, \tag{5.20}
\]

\[
\frac{W}{T} \frac{\partial w'}{\partial t'} + \frac{UW}{L} \left\{ u' \frac{\partial w'}{\partial x'} + v' \frac{\partial w'}{\partial y'} \right\} + \frac{W^2}{D} \left\{ w' \frac{\partial w'}{\partial z'} \right\} = -\frac{1}{\rho \rho'} \frac{P}{D} \frac{\partial p'}{\partial z'} - \tilde{g} \tilde{g}'. \tag{5.21}
\]
Consider the scaling analysis applied to the equation of mass conservation (4.7), this gives

\[
\frac{1}{T} \frac{\partial \rho'}{\partial t'} + \frac{U}{L} \left\{ u' \frac{\partial \rho'}{\partial x'} + v' \frac{\partial \rho'}{\partial y'} \right\} + \frac{W}{D} \frac{\partial \rho'}{\partial z'} + \rho' \frac{U}{L} \left\{ \frac{\partial u'}{\partial x'} + \frac{\partial v'}{\partial y'} \right\} + \rho' \frac{W}{D} \frac{\partial w'}{\partial z'} = 0. \tag{5.22}
\]

If the flow is incompressible then (5.22) reduces to

\[
\frac{U}{L} \left\{ \frac{\partial u'}{\partial x'} + \frac{\partial v'}{\partial y'} \right\} + \frac{W}{D} \frac{\partial w'}{\partial z'} = 0. \tag{5.23}
\]

This equation provides an upper bound on the scaling of the vertical velocity \( W \) such that

\[
W \leq O \left( \frac{UD}{L} \right), \tag{5.24}
\]

the reasoning behind this being an upper bound on \( W \) is that the vertical velocity can be smaller than (5.24) if there is cancellation between the two horizontal gradient terms in (5.23). Although this upper bound for \( W \) was derived from an incompressible perspective, the scaling is consistent with the corresponding compressible case. From the continuity bound (5.24) the scaling of the vertical derivatives in the horizontal momentum equations (5.19)-(5.20) is given by

\[
\frac{W U}{D} \leq O \left( \frac{U^2}{L} \right). \tag{5.25}
\]

Taking this upper bound as the correct scaling implies that all three terms in the advection part of the acceleration are of the same order, therefore the two horizontal components of the momentum equation are given by

\[
\begin{align*}
\frac{U}{T} \frac{\partial u'}{\partial t'} + \frac{U^2}{L} \left\{ u \frac{\partial u'}{\partial x'} + v \frac{\partial u'}{\partial y'} + w \frac{\partial u'}{\partial z'} \right\} - fU v' &= - \frac{P}{\partial L} \rho' \frac{\partial l'}{\partial x'}, \tag{5.26} \\
\frac{U}{T} \frac{\partial v'}{\partial t'} + \frac{U^2}{L} \left\{ u \frac{\partial v'}{\partial x'} + v \frac{\partial v'}{\partial y'} + w \frac{\partial v'}{\partial z'} \right\} + fU u' &= - \frac{P}{\partial L} \rho' \frac{\partial l'}{\partial y'}. \tag{5.27}
\end{align*}
\]
Chapter 5. The breakdown of the hydrostatic case

For the scaling of the pressure \( P \) the requirement is that the horizontal pressure gradients are of equal order to the acceleration terms therefore ensuring that the pressure acts as a forcing term otherwise the flow would be unaccelerated. This implies that

\[
P = \max \left( \rho L \left[ \frac{U}{T}, \frac{U^2}{L}, fU \right] \right), \tag{5.28}
\]

Therefore in considering the vertical momentum equation the two terms to compare are the vertical acceleration with the vertical pressure gradient. This implies

\[
\max \left( \frac{W}{T}, \frac{UW}{L}, \frac{W^2}{D} \right) = O \left( \frac{P}{D\rho} \right), \tag{5.29}
\]

using the upper bound of (5.25) and (5.28) substituted into (5.29) to give the order of the ratio of the vertical pressure gradient to the vertical acceleration namely

\[
\frac{\left( \frac{\partial \rho'}{\partial z'} \right)}{\left( \frac{D'\omega'}{Dt'} \right)} = O \left\{ \delta^2 \frac{\max \left[ \frac{1}{T}, \frac{U}{L} \right]}{\max \left[ \frac{1}{T}, \frac{U}{L}, f \right]} \right\}, \tag{5.30}
\]

where \( D'/Dt' \) is the non-dimensional material derivative operator and the parameter \( \delta \) is the aspect ratio and is defined as a ratio of the height to the depth of the fluid \((D/L)\). Also define the following two parameters

\[
\varepsilon_T = \frac{1}{TT}, \quad \varepsilon = \frac{U}{fL}, \tag{5.31}
\]

where \( \varepsilon_T \) and \( \varepsilon \) are the Rossby numbers and measure the relative importance of the local and convective accelerations respectively. If the Rossby number \( \varepsilon \) is \( O(1) \) or greater, then the ratio in (5.30) is of order \( \delta^2 \). If, however, \( \varepsilon \) is less than \( O(1) \) then the ratio is even smaller. Now, if the aspect ratio \( \delta \ll 1 \), then to at least \( O(\delta^2) \), the vertical acceleration is negligible and the vertical pressure gradient is

\[
\frac{\partial \rho'}{\partial z'} = -\rho' g' + O(\delta^2), \tag{5.32}
\]

dropping the \( O(\delta^2) \) term gives the hydrostatic approximation.
Chapter 5. The breakdown of the hydrostatic case

Returning to the scaling of the momentum equations, we are going to limit our attention to cases where both $\varepsilon_T$ and $\varepsilon$ are small. If the horizontal acceleration terms are neglected then for motion to take place in these two directions, there must be a balance between the Coriolis force and pressure gradient terms. This implies that

$$P = O(\rho f LU).$$

(5.33)

Therefore these results simplify (5.19)-(5.21) to

$$\varepsilon_T \frac{\partial u'}{\partial t'} + \varepsilon \left\{ u' \frac{\partial u'}{\partial x'} + v' \frac{\partial u'}{\partial y'} + w' \frac{\partial u'}{\partial z'} \right\} - v' = -\frac{1}{\rho'} \frac{\partial p'}{\partial x'},$$

(5.34)

$$\varepsilon_T \frac{\partial v'}{\partial t'} + \varepsilon \left\{ u' \frac{\partial v'}{\partial x'} + v' \frac{\partial v'}{\partial y'} + w' \frac{\partial v'}{\partial z'} \right\} + u' = -\frac{1}{\rho'} \frac{\partial p'}{\partial y'},$$

(5.35)

$$\varepsilon_T \delta^2 \frac{\partial w'}{\partial t'} + \varepsilon \delta^2 \left\{ u' \frac{\partial w'}{\partial x'} + v' \frac{\partial w'}{\partial y'} + w' \frac{\partial w'}{\partial z'} \right\} = -\frac{1}{\rho'} \frac{\partial p'}{\partial z'} - g'.$$

(5.36)

Turning our attention to

$$\frac{\varepsilon}{\varepsilon_T} = \frac{L}{UT},$$

(5.37)

if this ratio is large then the local time derivative dominates the nonlinear advection and so the equations are essentially linear. However, the nonlinear terms will be treated as equally important as the linear acceleration term, therefore the ratio (5.37) is set equal to 1 and so $\varepsilon = \varepsilon_T$. Adding this result to equations (5.34)-(5.36) gives the complete set of non-dimensionalised equations corresponding to the dimensionless forms (5.4)-(5.6)

$$\varepsilon \left\{ \frac{\partial u'}{\partial t'} + u' \frac{\partial u'}{\partial x'} + v' \frac{\partial u'}{\partial y'} + w' \frac{\partial u'}{\partial z'} \right\} - v' = -\frac{1}{\rho'} \frac{\partial p'}{\partial x'},$$

(5.38)

$$\varepsilon \left\{ \frac{\partial v'}{\partial t'} + u' \frac{\partial v'}{\partial x'} + v' \frac{\partial v'}{\partial y'} + w' \frac{\partial v'}{\partial z'} \right\} + u' = -\frac{1}{\rho'} \frac{\partial p'}{\partial y'},$$

(5.39)

$$\varepsilon \delta^2 \left\{ \frac{\partial w'}{\partial t'} + u' \frac{\partial w'}{\partial x'} + v' \frac{\partial w'}{\partial y'} + w' \frac{\partial w'}{\partial z'} \right\} = -\frac{1}{\rho'} \frac{\partial p'}{\partial z'} - g'.$$

(5.40)

together with the statement of mass conservation.
\[
\frac{\partial \rho'}{\partial t'} + u' \frac{\partial \rho'}{\partial x'} + v' \frac{\partial \rho'}{\partial y'} + w' \frac{\partial \rho'}{\partial z'} + \rho' \left( \frac{\partial u'}{\partial x'} + \frac{\partial v'}{\partial y'} + \frac{\partial w'}{\partial z'} \right) = 0. \tag{5.41}
\]

Instead of analysing these non-dimensional momentum equations (5.38)-(5.40), the corresponding non-dimensional vorticity equation is derived from the corresponding dimensional equation (4.16).

### 5.3 The non-dimensional vorticity equation

The non-dimensionalisation of the vorticity equation is done in exactly the same way as the momentum and continuity equations. In component form the \(i\)-th component of (4.16) is given by

\[
\left[ \frac{\partial}{\partial t'} + u' \frac{\partial}{\partial x'} + v' \frac{\partial}{\partial y'} + w' \frac{\partial}{\partial z'} \right] \left\{ \frac{1}{\rho} \left( \frac{\partial w'}{\partial y'} - \frac{\partial v'}{\partial z'} \right) \right\} \frac{\partial}{\partial x'} + \left\{ \frac{1}{\rho} \left( \frac{\partial u'}{\partial z'} - \frac{\partial w'}{\partial x'} \right) \right\} \frac{\partial}{\partial y'} + \left\{ \frac{1}{\rho} \left( \frac{\partial v'}{\partial x'} - \frac{\partial u'}{\partial y'} + f' \right) \right\} \frac{\partial}{\partial z'} u' + 1 + \frac{\partial (\rho', p')}{\partial (y', z')} = 0.
\tag{5.42}
\]

where the Jacobian term \(\partial (\rho, p) / \partial (y, z)\) is

\[
\frac{\partial (\rho, p)}{\partial (y, z)} = \frac{\partial \rho}{\partial y} \frac{\partial p}{\partial z} - \frac{\partial \rho}{\partial z} \frac{\partial p}{\partial y}.
\tag{5.43}
\]

Using the scalings and results derived in the previous section, the \(i\)-th component of the vorticity equation in non-dimensional form is given by

\[
\frac{D'}{Dt'} \left[ \frac{1}{\rho'} \left( \varepsilon \delta^2 \frac{\partial w'}{\partial y'} - \varepsilon \frac{\partial u'}{\partial z'} \right) \right] = \frac{1}{\rho'} \left\{ \left( \varepsilon \delta^2 \frac{\partial w'}{\partial y'} - \varepsilon \frac{\partial u'}{\partial z'} \right) \frac{\partial}{\partial x'} + \left( \varepsilon \frac{\partial u'}{\partial z'} - \varepsilon \delta^2 \frac{\partial w'}{\partial x'} \right) \frac{\partial}{\partial y'} \right\} \frac{\partial}{\partial x'} + \left( 1 + \varepsilon \frac{\partial u'}{\partial x'} - \varepsilon \frac{\partial u'}{\partial y'} \right) \frac{\partial}{\partial z'} \left( 1 + \frac{1}{\rho'^3} \frac{\partial (\rho', p')}{\partial (y', z')} \right).
\tag{5.44}
\]

Here the material derivative operator \(D'/Dt'\) is given by

\[
\frac{D'}{Dt'} = \frac{\partial}{\partial t'} + \mathbf{u'} \cdot \nabla' = \frac{\partial}{\partial t'} + u' \frac{\partial}{\partial x'} + v' \frac{\partial}{\partial y'} + w' \frac{\partial}{\partial z'}.
\tag{5.45}
\]
Stating the results for the other two components, which are of use in the later analysis, we find

\[
\frac{D'}{Dt'} \left[ 1 + \frac{1}{\rho'} \left( \frac{\varepsilon}{\rho'} \frac{\partial u'}{\partial x'} - \frac{\varepsilon}{\rho'} \frac{\partial w'}{\partial y'} \right) \right] = \frac{1}{\rho'} \left\{ \left( \frac{\varepsilon}{\rho'} \frac{\partial w'}{\partial y'} - \frac{\varepsilon}{\rho'} \frac{\partial v'}{\partial z'} \right) \frac{\partial}{\partial x'} + \left( \frac{\varepsilon}{\rho'} \frac{\partial u'}{\partial z'} - \frac{\varepsilon}{\rho'} \frac{\partial w'}{\partial x'} \right) \frac{\partial}{\partial y'} \right\} u' + \frac{1}{\rho'^3} \frac{\partial (\rho', p')}{\partial (x', y')}, \tag{5.48}\]

\[
\frac{D'}{Dt'} \left[ \frac{1}{\rho'} \left( 1 + \frac{1}{\rho'} \frac{\partial v'}{\partial x'} - \frac{\varepsilon}{\rho'} \frac{\partial u'}{\partial y'} \right) \right] = \frac{1}{\rho'} \left\{ \left( \frac{\varepsilon}{\rho'} \frac{\partial w'}{\partial y'} - \frac{\varepsilon}{\rho'} \frac{\partial v'}{\partial z'} \right) \frac{\partial}{\partial x'} + \left( \frac{\varepsilon}{\rho'} \frac{\partial u'}{\partial z'} - \frac{\varepsilon}{\rho'} \frac{\partial w'}{\partial x'} \right) \frac{\partial}{\partial y'} \right\} w' + \frac{1}{\rho'^3} \frac{\partial (\rho', p')}{\partial (x', y')}. \tag{5.49}\]

Combining equations (5.45)-(5.47) gives the following non-dimensional vorticity vector equation

\[
\frac{D'\xi'}{Dt'} = \left[ \frac{\xi'}{\rho'} \cdot \nabla' \right] u' + \frac{1}{\rho'^3} \nabla' \times \nabla' p', \tag{5.48}\]

where

\[
\xi' = \left( \frac{\varepsilon}{\rho'} \frac{\partial w'}{\partial y'} - \frac{\varepsilon}{\rho'} \frac{\partial v'}{\partial z'}, \frac{\varepsilon}{\rho'} \frac{\partial u'}{\partial z'} - \frac{\varepsilon}{\rho'} \frac{\partial w'}{\partial x'}, 1 + \frac{\varepsilon}{\rho'} \frac{\partial v'}{\partial x'} - \frac{\varepsilon}{\rho'} \frac{\partial u'}{\partial y'} \right). \tag{5.49}\]

The case of hydrostatic balance can easily be incorporated into the above set of equations by considering the limit \( \delta \to 0 \). It is interesting to note the similarity with the dimensional form of the limit given in equations (5.13)-(5.14). It is now possible to consider the hydrostatic limit applied to a number of different flow regimes, and to explain these results in the context of results seen in previous chapters. In the next few sections the flows and equations under consideration will be in non-dimensional form, therefore the dashes will be dropped. Any time that either a result is needed or referenced with respect to the corresponding dimensional form an explicit mention will be made.
5.4 Hydrostatic balance for a constant density and barotropic fluid

Our ultimate aim is to see the effect of adding the constraint of hydrostatic balance to the quaternionic structure of the equations of motion. However, before this can be done an understanding is needed of these non-dimensional variables, and so it is logical to find relationships or balance conditions between terms of like order in the Rossby number and aspect ratio. For small values of $\varepsilon$ and $\delta$, the velocity and pressure variables are expanded in terms of these parameters. In equations (5.38)-(5.40) and (5.45)-(5.47) only integer powers of $\varepsilon$ and $\delta$ appear and so a reasonable asymptotic expansion would be the following Taylor series expansion in terms of $\varepsilon, \delta$

\begin{align}
  u &= u_0(x, t) + \varepsilon u_1(x, t) + \delta \tilde{u}_1(x, t) + O(\varepsilon^2, \delta^2), \\
  v &= v_0(x, t) + \varepsilon v_1(x, t) + \delta \tilde{v}_1(x, t) + O(\varepsilon^2, \delta^2), \\
  w &= w_0(x, t) + \varepsilon w_1(x, t) + \delta \tilde{w}_1(x, t) + O(\varepsilon^2, \delta^2), \\
  p &= p_0(x, t) + \varepsilon p_1(x, t) + \delta \tilde{p}_1(x, t) + O(\varepsilon^2, \delta^2),
\end{align}

where $O(\varepsilon^2, \delta^2)$ represents higher order terms in both the Rossby number $\varepsilon$ and the aspect ratio $\delta$.

5.4.1 Constant density case

Consider the flow of a constant density fluid. Substituting the above expanded variables into (5.38)-(5.41) gives the following $O(1)$ terms
Chapter 5. The breakdown of the hydrostatic case

\[ v_0 = \frac{1}{\rho} \frac{\partial p_0}{\partial x}, \quad (5.54) \]

\[ u_0 = -\frac{1}{\rho} \frac{\partial p_0}{\partial y}, \quad (5.55) \]

\[ 0 = -\frac{1}{\rho} \frac{\partial p_0}{\partial z} - g, \quad (5.56) \]

\[ \frac{\partial u_0}{\partial x} + \frac{\partial v_0}{\partial y} + \frac{\partial w_0}{\partial z} = 0; \quad (5.57) \]

the terms (5.54) and (5.55) are known as the geostrophic approximation to the full horizontal momentum equations. In this leading order approximation the horizontal velocities reduce to a balance between the horizontal components of the Coriolis acceleration and the horizontal pressure gradients. These velocities \((u_0, v_0)\), usually denoted by \((u_g, v_g)\), are called the geostrophic velocity. In vector form they can be expressed as

\[ u_0 = \frac{1}{\rho} k \times \nabla p_0. \quad (5.58) \]

The third \(O(1)\) term says that for small \(\delta\) the vertical motion is in hydrostatic balance, this equation can be integrated to give

\[ p_0 = -\int \rho g dz = -\rho gz + f(x, y, t), \quad (5.59) \]

where \(f(x, y, t)\) is some arbitrary function of the horizontal displacement and time and could be determined by suitable initial or boundary conditions. Substituting the geostrophic relations (5.54) and (5.55) into the \(O(1)\) incompressibility constraint (5.57) implies that \(w_0\) is independent of the height \(z\). Finally, substituting the expression for \(p_0\) into the two geostrophic relations implies that the horizontal geostrophic velocities are also independent of height. The leading order flow is both hydrostatic and geostrophic and leads to the classical problem of the inability of the geostrophic approximation alone to determine \(p_0\) and hence \(u_0\) and \(v_0\). Considering terms of higher order, then at \(O(\varepsilon)\)
\[ \frac{\partial u_0}{\partial t} + u_0 \frac{\partial u_0}{\partial x} + v_0 \frac{\partial u_0}{\partial y} - v_1 = -\frac{1}{\rho} \frac{\partial p_1}{\partial x}, \quad (5.60) \]
\[ \frac{\partial v_0}{\partial t} + u_0 \frac{\partial v_0}{\partial x} + v_0 \frac{\partial v_0}{\partial y} + u_1 = -\frac{1}{\rho} \frac{\partial p_1}{\partial y}, \quad (5.61) \]
\[ 0 = \frac{\partial p_1}{\partial z}, \quad (5.62) \]
\[ \frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} + \frac{\partial w_1}{\partial z} = 0. \quad (5.63) \]

From the vertical component (5.62), the \( O(\varepsilon) \) pressure field is independent of height and together with the fact the geostrophic velocity is also height independent, the horizontal momentum equations (5.60) and (5.61) imply that the \( O(\varepsilon) \) horizontal velocities \((u_1, v_1)\) are independent of \( z \). It is worth noting that these velocities are not geostrophic and hence \( w_1 \) is not height independent. The departure of these velocities from balance with the \( O(\varepsilon) \) pressure field are due entirely to the acceleration of the leading order, \( O(1) \), velocity fields. The pressure terms in the horizontal momentum equations can be eliminated by considering \( \frac{\partial}{\partial x}(5.61) - \frac{\partial}{\partial y}(5.60) \) to obtain
\[ \frac{\partial \xi_0}{\partial t} + u_0 \frac{\partial \xi_0}{\partial x} + v_0 \frac{\partial \xi_0}{\partial y} = -\left( \frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} \right) = \frac{\partial w_1}{\partial z}, \quad (5.64) \]
where \( \xi_0 \) is the first order relative vertical vorticity
\[ \xi_0 = \frac{\partial v_0}{\partial x} - \frac{\partial u_0}{\partial y} = \frac{\partial^2 p_0}{\partial x^2} + \frac{\partial^2 p_0}{\partial y^2}. \quad (5.65) \]

To order \( O(\varepsilon) \) the rate of change of the relative vertical vorticity is equal to the convergence presence in the \( O(\varepsilon) \) ageostrophic field. Furthermore, the order \( \varepsilon \) relative vorticity \( \xi_1 \) takes the form
\[ \xi_1 = \frac{\partial v_1}{\partial x} - \frac{\partial u_1}{\partial y} = \frac{\partial^2 p_1}{\partial x^2} + \frac{\partial^2 p_1}{\partial y^2} + 2 \left( \frac{\partial^2 p_0}{\partial y \partial x} \right)^2 - 2 \frac{\partial^2 p_0}{\partial x^2} \frac{\partial^2 p_0}{\partial y^2}. \quad (5.66) \]

At this juncture it is worth considering the question (which has a fundamental effect on the form that the vorticity equation for a constant density fluid takes): what
is the dependence of the horizontal velocities on the height displacement $z$? In this section it has been shown that the geostrophic velocity and the $O(\varepsilon)$ velocities are independent of height. However, from consideration of (5.40), if the aspect ratio is suitable scaled to some order of the Rossby number then at some higher order there will be a balance between the vertical acceleration and the vertical pressure gradient. However, if the flow is in hydrostatic balance then only at leading order is the pressure dependent on the height and for all orders the horizontal velocities are independent of height. In fact, if under the assumption that the flow is incompressible and of constant density and that the aspect ratio $\delta \to 0$, then defining $h(x, y, t)$ and $h_{\text{bot}}(x, y)$ to be the height of the fluid above some reference level and the height of an arbitrary fixed rigid bottom respectively then (5.62) becomes an expression for the total pressure $p$ and

$$p = p_{\text{ref}} + \rho g (h - z), \quad (5.67)$$

where $p_{\text{ref}}$ is the pressure at the surface. The horizontal momentum equations are then

$$\frac{\partial u^*}{\partial t^*} + u^* \frac{\partial u^*}{\partial x^*} + v^* \frac{\partial u^*}{\partial y^*} - f v^* = -g^* \frac{\partial h^*}{\partial x^*}, \quad (5.68)$$

$$\frac{\partial v^*}{\partial t^*} + u^* \frac{\partial v^*}{\partial x^*} + v^* \frac{\partial v^*}{\partial y^*} + f u^* = -g^* \frac{\partial h^*}{\partial y^*}, \quad (5.69)$$

where the superscript $*$ indicates that the variables are in their dimensional form. The incompressibility constraint can now be integrated in $z$, as the horizontal velocities are independent of $z$, to give an explicit expression for the vertical velocity $w$, this in turn can be transformed into an equation for the depth of the fluid $h$ and

$$\frac{\partial H^*}{\partial t^*} + (v^* \cdot \nabla^*) H^* + H^* \left( \frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} \right) = 0, \quad (5.70)$$

where $H^* = h^* - h_{\text{bot}}^*$, $v^* = (u^*, v^*)$ and $\nabla^* = (\partial / \partial x^*, \partial / \partial y^*)$. Equations (5.68)-(5.70) are the well known shallow-water equations (see Pedlosky (1987) or Salmon (1998) for a complete discussion on these equations). Before saying what
effect this has on our \((\alpha, \chi)\) variables let us turn our attention to the components of the non-dimensional vorticity equation (5.44), (5.46)-(5.47).

### 5.4.2 The barotropic case

Substituting the asymptotic expansions given in (5.50)-(5.56) into the individual components of the vorticity equation, at \(O(1)\) gives

\[
\frac{\partial u_0}{\partial z} + \frac{1}{\rho} \frac{\partial (\rho, p_0)}{\partial (y, z)} = 0, \quad (5.71)
\]

\[
\frac{\partial v_0}{\partial z} + \frac{1}{\rho} \frac{\partial (\rho, p_0)}{\partial (z, x)} = 0, \quad (5.72)
\]

\[
\frac{D_0}{D_{t_0}} \left( \frac{1}{\rho} \right) = \frac{1}{\rho} \frac{\partial u_0}{\partial z} + \frac{1}{\rho^2} \frac{\partial (\rho, p_0)}{\partial (x, y)}, \quad (5.73)
\]

where the density is a function of its leading order dependent terms. The vertical component (5.73) can be simplified using the statement of mass conservation (5.41) to give

\[
\frac{\partial u_0}{\partial x} + \frac{\partial v_0}{\partial y} = \frac{1}{\rho} \frac{\partial (\rho, p_0)}{\partial (x, y)}. \quad (5.74)
\]

Equations (5.71) and (5.72) are the thermal wind relations, the name comes from the fact that density variations are usually connected with the winds or temperature fluctuations. The third term (5.74) gives an explicit expression for the horizontal geostrophic divergence in terms of the horizontal density-pressure gradients. If the flow is of constant density then (5.71)-(5.73) simplify to the results derived earlier, the leading order horizontal and vertical velocities are independent of height. In fact, this result holds for a barotropic fluid, regardless if the flow is incompressible or not. In fact, the geostrophic velocity is height independent for a barotropic fluid. By careful consideration of equations (5.44) and (5.46) it can be shown that under the constraint of hydrostatic balance the horizontal components of velocity are independent of height for a barotropic fluid. This can be proved using strong induction.
Let \( p(n) \) be the statement that for a barotropic flow in hydrostatic balance, to \( O(\varepsilon^n) \) the horizontal velocities are independent of height \( \forall n \). For \( p(0) \), the leading order expansion, the statement is true i.e.

\[
p(0) : \quad \frac{\partial u_0}{\partial z} = \frac{\partial v_0}{\partial z} = 0,
\]

(5.75)

now assume the statement is true for all orders up to and including \( n = k \) i.e.

\[
p(1 \rightarrow k) : \quad \frac{\partial}{\partial z} (u_1, v_1) = \cdots = \frac{\partial}{\partial z} (u_k, v_k) = 0,
\]

(5.76)

the aim is to prove for the case \( n = k + 1 \). Using these results and substituting the expansions into (5.44) without the baroclinic term and with the hydrostatic limit in place

\[
\left[ \frac{\partial}{\partial t} + \left( u_0 + \cdots + \varepsilon^{k+1} u_{k+1} \right) \frac{\partial}{\partial x} + \left( v_0 + \cdots + \varepsilon^{k+1} v_{k+1} \right) \frac{\partial}{\partial y} + \left( w_0 + \cdots + \varepsilon^{k+1} w_{k+1} \right) \frac{\partial}{\partial z} \right] \left\{ \frac{\varepsilon^{k+2}}{\rho (p_0 + \cdots)} \frac{\partial v_{k+1}}{\partial z} - \frac{1}{\rho (p_0 + \cdots)} \right\} = 0.
\]

(5.77)

grouping together terms of order \( k + 1 \), the result for \( p(k + 1) \) is

\[
p(k + 1) : \quad 0 = \frac{\partial u_{k+1}}{\partial z}.
\]

(5.78)

The corresponding result of \( v_{k+1} \) being independent of \( z \) is proved using the same method but instead substituting the expanded variables into (5.46). Therefore assuming true for \( n = 1 \rightarrow k \) leads to the result that the statement is true for \( n = k + 1 \), together with the fact that the case \( n = 0 \) is true means that the statement is true for all \( n \).

Hence for a barotropic flow, under the assumption of hydrostatic balance, the horizontal components of the velocity are independent of height. Together with the result derived in (5.14), the vorticity is therefore purely in the vertical direction and
is given by $\zeta = f + \partial v/\partial x - \partial u/\partial y$. This is also the corresponding result for the constant density case. The vorticity stretching vector is given by

$$\boldsymbol{\sigma} = (\boldsymbol{\xi} \cdot \nabla) \boldsymbol{u} = \begin{pmatrix} 0, 0, \zeta \frac{\partial w}{\partial z} \end{pmatrix}, \quad (5.79)$$

which is also strictly in the vertical component. This means that for these two particular cases, constant density and barotropic flows in hydrostatic balance, there is perfect alignment between the vorticity and vorticity stretching vector components and hence the vortex alignment vector is zero, i.e. $\chi = 0$. So the 4-vector $\mathbf{q}$ becomes a single scalar equation for the stretching rate. The breakdown of the quaternionic structure in the hydrostatic limit is summarised as follows: (1) for the case of a constant density fluid there are no terms to balance with the varying pressure in the vertical direction, after a leading order balance between the pressure gradient and the gravity, hence the horizontal velocities do not vary with respect to height. Similarly for (ii) a barotropic fluid in which the density depends only on the pressure, the corresponding thermal wind equation states that the geostrophic wind is independent of height and once again through an asymptotic analysis the vorticity and vorticity stretching terms are parallel.

Returning to our consideration of the Shallow-water equations (5.71)-(5.72) it is possible to conclude that there is no apparent quaternionic structure when considering the corresponding vorticity and its evolution as a framework for such a structure. The shallow-water momentum equations (5.71) and (5.72) can be combined (by taking the 2D curl) to give

$$\nabla_h \times \left( \frac{\partial \mathbf{v}}{\partial t} \right) + \nabla_h \times (\mathbf{k} \times \zeta \mathbf{v}) = 0, \quad (5.80)$$

where $\nabla_h = (\partial/\partial x, \partial/\partial y)$ and the variables are no longer non-dimensional. Together with the governing equation for the height of the fluid $H$, (5.80) simplifies to

$$\frac{D_h}{Dt} \left( \frac{\zeta}{H} \right) = 0, \quad (5.81)$$
where \( D_h/Dt = \partial/\partial t + v \cdot \nabla_h \) and the quantity \( \zeta/H \) is conserved. An alternative way of expressing the vorticity equation is given by

\[
\frac{D_h \zeta}{Dt} = -\zeta \nabla_h \cdot v. \tag{5.82}
\]

In this form, certain statements can be made regarding the corresponding \((\alpha - \chi)\) variables for this particular fluid flow. The vorticity stretching vector is given by \(-\zeta \nabla_h \cdot v k\) and therefore the stretching rate corresponding to the shallow-water equations is given by

\[
\alpha = -\nabla_h \cdot v, \tag{5.83}
\]

this is the negative horizontal divergence, for \(\nabla_h \cdot v < 0\) there is vortex stretching and for \(\nabla_h \cdot v > 0\) there is vortex compression. Trying to form the corresponding alignment vector \(\chi\) gives

\[
\chi = -\frac{\zeta k \times \zeta \nabla \cdot v k}{\zeta \cdot \zeta} = 0, \tag{5.84}
\]

the shallow-water equations lead to the trivial case of the corresponding vortex alignment vector being zero. Of course, this was expected as the shallow water equations are in essence hydrostatic, constant density flow. In fact, if \(\chi\) had been non-zero then due to the quasi-two dimensional appearance of the governing equations the alignment vector would have been in fact a scalar.

This result for the shallow-water equations may in fact suggest to some that the general theory that was defined only for fully three-dimensional systems in chapter 2 can not, in fact, be applied to any two-dimensional flow regimes. In fact, the theory can be applied to two-dimensional systems in cases in which there is a directly analogy between the three-dimensional vorticity and a corresponding key quantity in the prescribed two-dimensional flow. Let us now consider such a 2D flow. The discussion begins with an introduction to the theory of active scalars.
5.5 The two-dimensional quasi-geostrophic thermal active scalar

Active scalars (see Constantin (1994)) are in fact solutions to a particular class of equations that cover a substantial area in the study of two-dimensional, incompressible fluid flow. The particular type of scalar discussed in this section will be the simplest case, that is ones that are unchanged or invariant to changes in the original equations of motion. Both passive and active scalars are solutions of advection-diffusion equations with given non-divergent velocities of the form

\[
\frac{\partial \theta}{\partial t} + v \cdot \nabla \theta - \mu \Delta \theta = f, \tag{5.85}
\]

where \( \mu \) is the diffusion coefficient and \( f \) represents the forcing term. The main difference between the two types of scalars is that the active ones determine their own velocities

\[
v = \nabla^\perp \psi, \tag{5.86}
\]

where \( \psi \) is the corresponding stream-function and \( \nabla^\perp \) represents the curl operator and is defined as

\[
\nabla^\perp = J \nabla, \tag{5.87}
\]

where \( J \) is a 2 \( \times \) 2 matrix given by

\[
J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{5.88}
\]

The stream-function \( \psi \) is given by

\[
\psi = A(\theta), \tag{5.89}
\]

this equation (5.89) is the equation of state and \( A \) is some non-local operator that defines the stream-function in terms of the active scalar \( \theta \). The most familiar (and
probably most important) example of a two-dimensional, incompressible system that can be expressed as an active scalar are the Navier-Stokes equations. In this case the scalar $\theta$ is in fact the vorticity $\omega$. There are other practical and significant active scalar equations. The one that will be considered now is based on a model of quasi-geostrophic flow.

The equation of motion that are considered are as follows

$$\frac{D_h \theta}{Dt} = \frac{\partial \theta}{\partial t} + (\mathbf{v} \cdot \nabla_h) \theta = 0,$$

(5.90) the variable $\theta$ is conserved and represents the potential temperature. This equation is equivalent to (5.85) in the absence of diffusion and forcing. The two-dimensional velocity $\mathbf{v}$ is incompressible and therefore a stream function $\psi$ can be constructed as follows

$$\mathbf{v} = \nabla \psi = \left( -\frac{\partial \psi}{\partial y}, \frac{\partial \psi}{\partial x} \right),$$

(5.91) so $\mathbf{v}$ is the fluid velocity and the stream function $\psi$ can be identified with the corresponding pressure. For completeness, the stream function satisfies

$$(-\Delta)^{\frac{1}{2}} \psi = -\theta,$$

(5.92) where the operator $(-\Delta)^{\frac{1}{2}}$ is determined by the Fourier transform

$$\hat{\psi}(\mathbf{k}) = \int e^{2\pi i \mathbf{x} \cdot \mathbf{k}} \hat{\psi}(\mathbf{k}) \, d\mathbf{k},$$

(5.93) where $\mathbf{x} = (x, y)$ and therefore

$$(-\Delta)^{\frac{1}{2}} \psi = 2\pi \int e^{2\pi i \mathbf{x} \cdot \mathbf{k}} |\mathbf{k}| \hat{\psi}(\mathbf{k}) \, d\mathbf{k}.$$  

(5.94) These equations (5.90)-(5.92) are derived from the more general quasigeostrophic approximation (Pedlosky (1987)) for fluid flow in a three-dimensional half-space that is rapidly rotating and has both small Ekman and Rossby numbers. For the case of solutions with constant potential vorticity in the flow and constant buoyancy frequency the general quasi-geostrophic equations reduce to equations for the
temperature on the two-dimensional boundary given in (5.90)-(5.92). The statistical turbulence theory for these special quasigeostrophic flows have been studied by Blumen (1978) and Pierrehumbert et al. (1994), furthermore, some qualitative features of the solution to these equations (in a geophysical context) are discussed in Held et al. (1995).

Recall that in the previous section mention was given of the general theory of Chapter 2 being applicable to a two-dimensional system in which there exists a direct physical and mathematical analogue between such a two-dimensional system and the three-dimensional Euler equations. This analogy between the two-dimensional quasi-geostrophic active scalar and Euler (see Constantin et al. (1994)) begins by introducing

$$\nabla^\perp \theta = \left( -\frac{\partial \theta}{\partial y}, \frac{\partial \theta}{\partial x} \right). \quad (5.95)$$

It is the role of the vector field $\nabla^\perp \theta$ for the 2-D QG active scalar that is completely analogous to the vorticity $\omega$ in 3-D incompressible Euler. A similar structure, if not slightly superficial, is seen on differentiating equation (5.90) to give the following evolution equation for $\nabla^\perp \theta$

$$\frac{D}{Dt} (\nabla^\perp \theta) = (\nabla v) \nabla^\perp \theta, \quad (5.96)$$

this equation (5.96) resembles the equation for the vorticity for the three-dimensional incompressible Euler equations (3.8).

However, the analogy between the two extends further to detailed analytic and geometric properties. The one considered here, as it has a direct bearing on later work, will be the study of the geometric properties. From the evolution equation for $\theta$ in (5.90) it follows that the level sets, $\theta = \text{constant}$, follow the flow of the fluid and the quantity $\nabla^\perp \theta$ is tangent to those level sets. Recall that the vorticity in three-dimensional incompressible Euler is tangent to vortex lines (that further move with the flow itself). Thus the level sets of $\theta$ are analogous to the vortex lines for the three-dimensional Euler equations. The quantity $|\nabla^\perp \theta|$ is the infinitesimal length
Chapter 5. The breakdown of the hydrostatic case

of a level set of $\theta$ for the two-dimensional quasi-geostrophic active scalar, and its evolution equation is given by

$$\frac{D}{Dt}|\nabla^\perp \theta| = \alpha |\nabla^\perp \theta|,$$  \hspace{1cm} (5.97)

where $\alpha$ is given explicitly by

$$\alpha = \frac{\nabla^\perp \theta \cdot (\nabla \mathbf{v}) \nabla^\perp \theta}{\nabla^\perp \theta \cdot \nabla^\perp \theta} = \eta \cdot S \eta,$$  \hspace{1cm} (5.98)

where $\eta = \nabla^\perp \theta / |\nabla^\perp \theta|$ and $S$ is the strain matrix $\frac{1}{2} (v_{i,j} + v_{j,i})$. Notice the similarity between the evolution of the quantity $|\nabla^\perp \theta|$ and the corresponding equation for the stretching rate with respect to the same equations for the vorticity magnitude $\omega$ for the three dimensional Euler vorticity equation and its corresponding equation for the stretching rate. It is clear that there is a geometric analogue between the two different dimensional systems. It is now possible, in a way which has been seen in previous chapters, to construct a corresponding alignment scalar $\chi$; it is no longer a $\chi$-vector as the corresponding flow is only in the two spatial dimensions. This alignment scalar $\chi$ is given by

$$\chi = \frac{\nabla^\perp \theta \times (\nabla \mathbf{v}) \nabla^\perp \theta}{\nabla^\perp \theta \cdot \nabla^\perp \theta} = \eta \times S \eta.$$  \hspace{1cm} (5.99)

The equations for the evolution of these two quantities can be quoted using the results obtained in Chapter 2

$$\frac{D_h \alpha}{Dt} = \chi^2 - \alpha^2 + \frac{\nabla^\perp \theta \cdot \frac{D_h}{Dt} [(\nabla \mathbf{v}) (\nabla^\perp \theta)]}{\nabla^\perp \theta \cdot \nabla^\perp \theta},$$  \hspace{1cm} (5.100)

$$\frac{D_h \chi}{Dt} = -2\chi \alpha + \frac{\nabla^\perp \theta \times \frac{D_h}{Dt} [(\nabla \mathbf{v}) (\nabla^\perp \theta)]}{\nabla^\perp \theta \cdot \nabla^\perp \theta}.$$  \hspace{1cm} (5.101)

To calculate the evolution term $\frac{D_h}{Dt} [(\nabla \mathbf{v}) (\nabla^\perp \theta)]$ the Ertel result (3.27) still holds and its two-dimensional analogue is given by

$$\frac{D_h}{Dt} [(\nabla \mathbf{v}) (\nabla^\perp \theta)] = (\nabla^\perp \theta \cdot \nabla) \frac{D_h \mathbf{v}}{Dt}.$$  \hspace{1cm} (5.102)
Chapter 5. The breakdown of the hydrostatic case

The only way to evaluate the right-hand side of (5.102) is to apply the early results for the horizontal velocities in terms of the stream function \( \psi \) given in (5.91) and using the following expression for the material derivative in terms of \( \psi \)

\[
\frac{Dh}{Dt} = \partial \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla_h = \frac{\partial}{\partial t} - \frac{\partial \psi}{\partial y} \frac{\partial}{\partial x} + \frac{\partial \psi}{\partial x} \frac{\partial}{\partial y}
\]

the expression in equation (5.102) is given by the square matrix

\[
\nabla_h \left( \frac{Dh\mathbf{v}}{Dt} \right) \nabla \perp \theta = \begin{pmatrix}
-\frac{\partial}{\partial x} \left\{ \psi_{yt} - \psi_y \psi_{xy} + \psi_x \psi_{yy} \right\} & \frac{\partial}{\partial x} \left\{ \psi_{xt} - \psi_y \psi_{xx} + \psi_x \psi_{xy} \right\} \\
-\frac{\partial}{\partial y} \left\{ \psi_{yt} - \psi_y \psi_{xy} + \psi_x \psi_{yy} \right\} & \frac{\partial}{\partial y} \left\{ \psi_{xt} - \psi_y \psi_{xx} + \psi_x \psi_{xy} \right\}
\end{pmatrix} \nabla \perp \theta.
\]

One of the key differences to Euler is that, due to the way that the variables and problem have been defined, there is no explicit prognostic equation for the horizontal velocities and the only re-course is to directly substitute in terms of the known stream-function \( \psi \). Of course, the real aim of this section was to make the reader aware that not all two-dimensional systems are trivial in the context of the (three-dimensional) general theory of Chapter 2. In fact, the theory can be applied to a two-dimensional system when there is a direct analytic and geometric analogy between the three-dimensional vorticity and a corresponding quantity in the prescribed two-dimensional flow.

### 5.6 Summary

This chapter has highlighted some of the main problems of the general theory of Chapter 2. First, the vorticity equation describing an approximated set of equations to the full Euler equations will begin to exhibit non-vortex stretching terms and so Ertel’s theorem has to be modified to take this into account. Secondly, there is the problem of applying a variant of this Ertel result when not all variables have a prognostic equation, for example, the vertical co-ordinate in a situation of hydrostatic balance. The solution to this problem is to take the limit of the aspect ratio tending to zero after the second material derivative of the vorticity is found. This
is possible by considering the non-dimensional form of the equations of motion. Finally, the breakdown of the quaternionic structure, when the alignment vector is zero, is due to the perfect alignment of the vorticity with the vortex stretching terms which reduce the 4-vector $\mathbf{q}$ to a scalar equation for the stretching rate. Also, two different sets of (Shallow-water and the Quasi-geostrophic active scalar) equations were considered as examples of when it is possible (and under what conditions) for the general theory, which up until now had only been considered in the context of three-dimensional problems, can be applied to two-dimensional systems.
Chapter 6

The non-hydrostatic and hydrostatic, primitive equations

The aim now is to apply the results of the previous chapters to one particular system namely the primitive equations in their Boussinesq form. The momentum equations in their non-hydrostatic form and the corresponding vorticity equation are stated, and these equations will be non-dimensionalised. The Ertel result, which enables the closure of the corresponding stretching rate and alignment vector evolution equations, will be considered. The hydrostatic limit will then be applied to the equations and the non-dimensional form of $\alpha$ and $\chi$ will be derived at leading and higher orders. The full system will then be closed by deriving the constraint equation in its non-dimensional form.

6.1 Equations of motion

The primitive equations in their non-hydrostatic, Boussinesq form, see Hoskins (1975) and Cullen (2002), are given by
Chapter 6. The non-hydrostatic and hydrostatic, primitive equations

\[
\begin{align*}
\frac{Du}{Dt} - f v + \frac{\partial \phi}{\partial x} &= 0, \\
\frac{Dv}{Dt} + f u + \frac{\partial \phi}{\partial y} &= 0, \\
\frac{Dw}{Dt} - \frac{g}{\theta_r} \theta + \frac{\partial \phi}{\partial z} &= 0, \\
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} &= 0, \\
\frac{D\theta}{Dt} &= 0,
\end{align*}
\]

(6.1) \hspace{1cm} (6.2) \hspace{1cm} (6.3) \hspace{1cm} (6.4) \hspace{1cm} (6.5)

where \( \phi \) is the geopotential, \( \theta \) the potential temperature, \( \theta_r \) a constant reference potential temperature and the material derivative \( D/Dt = \partial/\partial t + u \partial/\partial x + v \partial/\partial y + w \partial/\partial z \). The corresponding vorticity equation can easily be derived by combining the advection term of the material derivative \((u \cdot \nabla u)\) with the Coriolis term \((f k \times u)\) and then taking the curl to give

\[
\frac{D\xi}{Dt} = (\xi \cdot \nabla) u - k \times \frac{g}{\theta_r} \nabla \theta,
\]

(6.6)

where the vorticity \( \xi \) is given by

\[
\xi = \left( \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} - \frac{\partial u}{\partial x}, f + \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right).
\]

(6.7)

To calculate the evolution of the right-hand side of equation (6.6) the Ertel result is no longer applicable. This is because the corresponding vorticity equation does not consist solely of a stretching term \((\xi \cdot \nabla u)\) but also incorporates a baroclinic term \(\left(k \times \frac{g}{\theta_r} \nabla \theta\right)\). Applying the result (5.17) gives the following expression for the evolution of the vorticity equation

\[
\frac{D}{Dt} \left( \xi \cdot \nabla u - k \times \frac{g}{\theta_r} \nabla \theta \right) = \frac{D}{Dt} \left( -k \times \frac{g}{\theta_r} \nabla \theta \right) - \left( k \times \frac{g}{\theta_r} \nabla \theta \cdot \nabla \right) u + (\xi \cdot \nabla) \frac{Du}{Dt}.
\]

(6.8)

The first term on the right-hand side of the above expression can be evaluated and simplified by using the result that the potential temperature is conserved and the
full form of the evolution of the vorticity, when the vector form of the momentum equation is substituted into equation (6.8), then

\[
\frac{D}{Dt} \left( \xi \cdot \nabla u - k \times \frac{g}{\theta_r} \nabla \theta \right) = \left( -\frac{\partial u}{\partial y} \cdot \nabla \theta, \frac{\partial u}{\partial x} \cdot \nabla \theta, 0 \right) - k \times \frac{g}{\theta_r} \nabla \cdot \nabla u + \nabla \left( \frac{g}{\theta_r} \theta k - \nabla \phi - f k \times u \right) \xi. \tag{6.9}
\]

This is the Ohkitani result for the non-hydrostatic, primitive equations. Recall, in Chapter 4, that when this result is substituted into the general quaternionic equation for \( q \) then it is advisable to re-write this expression in terms of previous defined 4-vectors. The author believes this is possible and would probably be an interesting problem for any future work. The evolution equations for the stretching rate and the alignment vector can now be closed with the above expression. However, to close the system a constraint equation, giving the relationship between the variables seen in the momentum and corresponding vorticity equation, needs to be derived. Taking the divergence of the momentum equations (6.1)-(6.3) and using the fact that the flow is non-divergent gives

\[
\frac{u_{j,i} u_{i,j}}{\theta_r} - f \left( \frac{\partial u}{\partial x} - \frac{\partial u}{\partial y} \right) - \frac{g}{\theta_r} \frac{\partial \theta}{\partial z} = -\Delta \phi. \tag{6.10}
\]

If the evolution equations for the stretching rate and alignment vector were now written in full, the equations would be quite complicated and the same could be said for the corresponding 4-vector (quaternionic) form. More useful and significant results can be derived when the non-dimensional form of the equations and their limits in terms of non-dimensional parameters are considered. Although the quaternionic relationship in these vorticity variables is now becoming very complicated as each successive approximation increases the number of physical/mathematical quantities, the quaternionic form of the momentum equations can still be written in an elegant and simple form.
6.1.1 The quaternionic form of the primitive equations

Before continuing to the main part of this chapter, which is to consider the effect of taking the hydrostatic limit in the primitive equation in Boussinesq form, the case of re-writing the momentum equations in quaternionic form is considered. The vector form of the momentum equations can be re-written as

\[
\frac{\partial \mathbf{u}}{\partial t} + (\nabla \times \mathbf{u} + f \mathbf{k}) \times \mathbf{u} = -\nabla \left( \phi + \frac{1}{2} \mathbf{u}^2 \right) + g \frac{\partial \theta}{\partial r} \mathbf{k}. \tag{6.11}
\]

By following the same procedure as in Chapter 3, (6.11) in quaternionic form is given by

\[
\frac{\partial \mathbf{U}}{\partial t} = \frac{1}{2} \left[ \mathbf{U} \otimes \mathbf{\tilde{w}} - \mathbf{\tilde{w}} \otimes \mathbf{U} \right] - \nabla \otimes \mathbf{\tilde{P}} + \mathbf{\tilde{\Theta}}, \tag{6.12}
\]

where \( \mathbf{\tilde{w}} = (0, \text{curl} \mathbf{u} + f \mathbf{k})^T \) and \( \mathbf{\tilde{\Theta}} = \left( 0, \frac{g}{r} \theta \mathbf{k} \right)^T \). The result of taking the quaternionic curl of (6.12) gives not only the vorticity equation (6.8) but the corresponding constraint equation (6.10).

6.2 Non-dimensional form of the primitive equations

As in the previous chapter, the easiest way of implementing hydrostatic balance is to consider the equations in their non-dimensional form. By considering the same form of scaling for the velocities and spatial variables and introducing new scalings for the geopotential \( \phi = \Phi \phi' \) and for the potential temperature \( \theta = \Theta \theta' \), equations (6.1)-(6.5) become
Chapter 6. The non-hydrostatic and hydrostatic, primitive equations

\[
\frac{U}{T} \frac{\partial u'}{\partial t'} + U^2 \frac{L}{\partial x'} + U \frac{\partial u'}{\partial x'} + U \frac{\partial u'}{\partial y'} + \frac{W}{D} \frac{\partial u'}{\partial z'} = -f U u' = -\frac{\Phi}{L} \frac{\partial \phi'}{\partial x'}, \tag{6.13}
\]

\[
\frac{U}{T} \frac{\partial v'}{\partial t'} + U^2 \frac{L}{\partial x'} + U \frac{\partial v'}{\partial x'} + U \frac{\partial v'}{\partial y'} + \frac{W}{D} \frac{\partial v'}{\partial z'} = -f U v' = -\frac{\Phi}{L} \frac{\partial \phi'}{\partial y'}, \tag{6.14}
\]

\[
\frac{W}{T} \frac{\partial w'}{\partial t'} + U W \frac{L}{\partial x'} + U \frac{\partial w'}{\partial x'} + U \frac{\partial w'}{\partial y'} + \frac{W}{D} \frac{\partial w'}{\partial z'} = -\frac{g}{\theta_r} \frac{\partial \theta'}{\partial z'} = -\frac{\Phi}{D} \frac{\partial \phi'}{\partial z'}, \tag{6.15}
\]

\[
\frac{U}{L} \left( \frac{\partial u'}{\partial x'} + \frac{\partial v'}{\partial y'} \right) + \frac{W}{D} \frac{\partial u'}{\partial z'} = 0, \tag{6.16}
\]

\[
\frac{1}{T} \frac{\partial \theta'}{\partial t'} + \frac{U}{L} \left( \frac{\partial u'}{\partial x'} + \frac{\partial v'}{\partial y'} \right) + \frac{W}{D} \frac{\partial \theta'}{\partial z'} = 0. \tag{6.17}
\]

Using the same arguments as in the previous chapters, the fully non-dimensional form of the momentum equations (6.1)-(6.3) are given by (without the dashes)

\[
\varepsilon \left\{ \frac{\partial u'}{\partial t'} + u \frac{\partial u'}{\partial x'} + v \frac{\partial u'}{\partial y'} + w \frac{\partial u'}{\partial z'} \right\} - v = -\frac{\partial \phi'}{\partial x'}, \tag{6.18}
\]

\[
\varepsilon \left\{ \frac{\partial v'}{\partial t'} + u \frac{\partial v'}{\partial x'} + v \frac{\partial v'}{\partial y'} + w \frac{\partial v'}{\partial z'} \right\} + u = -\frac{\partial \phi'}{\partial y'}, \tag{6.19}
\]

\[
\varepsilon^2 \left\{ \frac{\partial w'}{\partial t'} + u \frac{\partial w'}{\partial x'} + v \frac{\partial w'}{\partial y'} + w \frac{\partial w'}{\partial z'} \right\} - \theta = -\frac{\partial \phi'}{\partial z'}. \tag{6.20}
\]

The non-dimensional forms of the incompressibility constraint and statement of potential temperature conservation are the same as in the dimensional form. The hydrostatic limit can easily be implied by taking the limit of the aspect ratio to zero. The variables can now be asymptotically expanded in terms of the Rossby number

\[
(u, \phi, \theta) = (u_0, \phi_0, \theta_0) + \varepsilon (u_1, \phi_1, \theta_1) + O (\varepsilon^2), \tag{6.21}
\]

At leading order ($\varepsilon^0$), $u_0$ and $v_0$ are given by

\[
u_0 = \frac{\partial \phi_0}{\partial y}, \quad v_0 = \frac{\partial \phi_0}{\partial x}, \tag{6.22}
\]
and together with the vertical component (6.20) the following relationships for the change in the leading order horizontal velocities in the vertical direction are

$$\frac{\partial u_0}{\partial z} = -\frac{\partial \theta_0}{\partial y}, \quad \frac{\partial v_0}{\partial z} = \frac{\partial \theta_0}{\partial x}. \quad (6.23)$$

At leading order the expressions in (6.22) are the well-known geostrophic relations and combining with the hydrostatic relation gives the thermal wind relations (6.23). The thermal wind relations are one of the key components to the results of this chapter. Another result which will also be of assistance is the next order of terms. These are very similar to the results derived in the previous section and are stated here as

$$\frac{\partial u_0}{\partial t} + u_0 \frac{\partial u_0}{\partial x} + v_0 \frac{\partial u_0}{\partial y} - v_1 = -\frac{\partial \phi_1}{\partial x}, \quad (6.24)$$
$$\frac{\partial v_0}{\partial t} + u_0 \frac{\partial v_0}{\partial x} + v_0 \frac{\partial v_0}{\partial y} + u_1 = -\frac{\partial \phi_1}{\partial y}, \quad (6.25)$$
$$\theta_1 = \frac{\partial \phi_1}{\partial z}, \quad (6.26)$$
$$\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} + \frac{\partial w_1}{\partial z} = 0. \quad (6.27)$$

For the two cases, hydrostatic balance applied to both constant density and barotropic flow, of the previous chapter, the breakdown of the quaternionic structure was due to the change in the leading order horizontal velocities being zero in the vertical direction. For this system of equations this is no longer the case and the thermal wind relations are testament to that. The corresponding, non-hydrostatic, vorticity equation in non-dimensional form is given by

$$\frac{D\xi}{Dt} = (\xi \cdot \nabla)u - k \times \nabla \theta, \quad (6.28)$$

where the non-dimensional vorticity absolute $\xi$ is

$$\xi = \left( \varepsilon \delta^2 \frac{\partial w}{\partial y} - \varepsilon \frac{\partial v}{\partial z}, \varepsilon \frac{\partial u}{\partial z} - \varepsilon \delta^2 \frac{\partial w}{\partial x}, 1 + \varepsilon \frac{\partial v}{\partial x} - \varepsilon \frac{\partial u}{\partial y} \right). \quad (6.29)$$
Re-writing the right-hand side of equation (6.28) as the vector $\sigma$, applying the result in (6.9) for the evolution of the vorticity equation and considering each component of $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ in turn gives

\[
\frac{D\sigma_1}{Dt} = (\xi \cdot \nabla) \varepsilon^{-1} \left\{ v - \frac{\partial \phi}{\partial x} \right\} - \frac{\partial u}{\partial y} \cdot \nabla \theta - \mathcal{L} u, \tag{6.30}
\]

\[
\frac{D\sigma_2}{Dt} = - (\xi \cdot \nabla) \varepsilon^{-1} \left\{ u + \frac{\partial \phi}{\partial y} \right\} + \frac{\partial u}{\partial x} \cdot \nabla \theta - \mathcal{L} v, \tag{6.31}
\]

\[
\frac{D\sigma_3}{Dt} = (\xi \cdot \nabla) \varepsilon^{-1} \delta^{-2} \left\{ \theta - \frac{\partial \phi}{\partial z} \right\} - \mathcal{L} w, \tag{6.32}
\]

where the operator $\mathcal{L} = k \times \nabla \theta \cdot \nabla$. The fully non-dimensional form, in terms of the Rossby number and aspect ratio, of the stretching rate can be derived by substituting (6.30)-(6.32) into equation (2.18) to give

\[
\varepsilon \delta^2 \xi^2 \frac{D\alpha}{Dt} = \varepsilon \delta^2 \xi^2 (\chi^2 - \alpha^2) + \delta^2 \xi \left\{ \xi \cdot \nabla \left( v - \frac{\partial \phi}{\partial x} \right) - \varepsilon \frac{\partial u}{\partial y} \cdot \nabla \theta - \varepsilon \delta^2 \mathcal{L} u \right\}
\]

\[
- \delta^2 \xi \left\{ \xi \cdot \nabla \left( u + \frac{\partial \phi}{\partial y} \right) - \varepsilon \frac{\partial u}{\partial x} \cdot \nabla \theta + \varepsilon \mathcal{L} v \right\} + \xi \left\{ \xi \cdot \nabla \left( \theta - \frac{\partial \phi}{\partial z} \right) - \varepsilon \delta^2 \mathcal{L} w \right\}. \tag{6.33}
\]

For completeness, each term in the alignment vector is considered and let $\chi = (\chi_1, \chi_2, \chi_3)$ and so

\[
\varepsilon \delta^2 \xi^2 \frac{D\chi_1}{Dt} = - 2 \varepsilon \delta^2 \xi^2 \chi_1 \alpha + \xi \left\{ \xi \cdot \nabla \left\{ \theta - \frac{\partial \phi}{\partial z} \right\} - \varepsilon \delta^2 \frac{\partial u}{\partial y} \cdot \nabla \theta - \varepsilon \delta^2 \mathcal{L} u \right\}
\]

\[
+ \delta^2 \chi_3 \left\{ \xi \cdot \nabla \left\{ u + \frac{\partial \phi}{\partial y} \right\} - \varepsilon \frac{\partial u}{\partial x} \cdot \nabla \theta + \varepsilon \mathcal{L} v \right\}, \tag{6.34}
\]

\[
\varepsilon \delta^2 \xi^2 \frac{D\chi_2}{Dt} = - 2 \varepsilon \delta^2 \xi^2 \chi_2 \alpha + \delta^2 \chi_3 \left\{ \xi \cdot \nabla \left\{ v - \frac{\partial \phi}{\partial x} \right\} - \varepsilon \frac{\partial u}{\partial y} \cdot \nabla \theta - \varepsilon \mathcal{L} u \right\}
\]

\[
- \xi_1 \left\{ \xi \cdot \nabla \left\{ \theta - \frac{\partial \phi}{\partial z} \right\} - \varepsilon \delta^2 \mathcal{L} w \right\}, \tag{6.35}
\]

\[
\varepsilon \xi^2 \frac{D\chi_3}{Dt} = - 2 \varepsilon \xi^2 \chi_3 \alpha - \xi_1 \left\{ \xi \cdot \nabla \left\{ u + \frac{\partial \phi}{\partial y} \right\} - \varepsilon \frac{\partial u}{\partial x} \cdot \nabla \theta + \varepsilon \mathcal{L} v \right\}
\]

\[
- \xi_2 \left\{ \xi \cdot \nabla \left\{ v - \frac{\partial \phi}{\partial x} \right\} - \varepsilon \frac{\partial u}{\partial y} \cdot \nabla \theta - \varepsilon \mathcal{L} u \right\}. \tag{6.36}
\]
These equations are in non-dimensional form but a matching of like order terms is not possible until each variable is expanded asymptotically. In fact, the analysis starts by considering the expansion of the \((\alpha, \chi)\) variables. To achieve this the vorticity and \(\sigma\) vector must be expanded first. Applying the hydrostatic limit, and re-writing the vorticity and vorticity evolution vector in terms of the Rossby number using the asymptotic expressions given in (6.21), gives

\[
\xi = (0, 0, 1) + \varepsilon \left( -\frac{\partial v_0}{\partial z}, \frac{\partial u_0}{\partial x} - \frac{\partial u_0}{\partial y} \right) + O(\varepsilon^2), \quad (6.37)
\]

\[
\sigma = \left( \frac{\partial u_0}{\partial z} + \frac{\partial \theta_0}{\partial y}, \frac{\partial v_0}{\partial z} - \frac{\partial \theta_0}{\partial x}, 0 \right) + \varepsilon \left( \frac{\partial u_1}{\partial z} + \frac{\partial \theta_1}{\partial y} - \frac{\partial v_0}{\partial x} \frac{\partial u_0}{\partial z} + \frac{\partial u_0}{\partial z} \frac{\partial u_0}{\partial y} \right) + O(\varepsilon^2).
\]

The leading order \(O(1)\) term in \(\sigma\) is zero due to the thermal wind relations, and so the vorticity stretching vector \(\sigma\) is of order \(\varepsilon\). Recall that in the definitions of the stretching rate and the alignment vector both have a denominator equal to \(|\xi|^2\), this can be expanded using the binomial theorem (with the limitation that this expansion is only valid for small Rossby number \(\varepsilon << 1\)) and

\[
|\xi|^2 = 1 - 2\varepsilon \left( \frac{\partial v_0}{\partial x} - \frac{\partial u_0}{\partial y} \right) + O(\varepsilon^2).
\]

(6.39)

Combining these three results (6.37)-(6.41), gives the following non-dimensional, asymptotic expansion for the stretching rate

\[
\alpha = (\xi \cdot \sigma) |\xi|^{-2} = \varepsilon \frac{\partial w_1}{\partial z} + O(\varepsilon^2).
\]

(6.40)

Recall that equations (5.67)-(5.68), stated that the \(O(\varepsilon)\) change in vertical velocity with respect to height is equal to the \(O(1)\) rate of change of the relative vertical vorticity. Similarly, in component form, the alignment vector can also be calculated and for the first component \(\chi_1\) is given by

\[
\chi_1 = -\varepsilon \left( \frac{\partial v_1}{\partial z} + \frac{\partial u_0}{\partial z} \frac{\partial v_0}{\partial y} - \frac{\partial u_0}{\partial y} \frac{\partial v_0}{\partial z} - \frac{\partial \theta_1}{\partial x} \right) + O(\varepsilon^2),
\]

(6.41)
differentiating equation (6.24), the acceleration equation for the geostrophic velocity \( u_0 \), and substituting this result together with the hydrostatic balance relation into equation (6.41) gives

\[
\chi_1 = -\varepsilon \left\{ \frac{\partial}{\partial t} \left( \frac{\partial u_0}{\partial z} \right) + u_0 \frac{\partial}{\partial x} \left( \frac{\partial u_0}{\partial z} \right) + v_0 \frac{\partial}{\partial y} \left( \frac{\partial u_0}{\partial y} \right) \right\} + O(\varepsilon^2), \tag{6.42}
\]

finally, after applying the thermal wind equation, the first component of the alignment vector is at leading order \( \varepsilon \) and is given by

\[
\chi_1 = -\frac{D_0}{Dt} \left( \frac{\partial u_0}{\partial z} \right) = \frac{D_0}{Dt} \left( \frac{\partial \theta_0}{\partial y} \right) \tag{6.43}
\]

where \( D_0/Dt \) is the first order material derivative and is simply

\[
\frac{D_0}{Dt} = \frac{\partial}{\partial t} + u_0 \frac{\partial}{\partial x} + v_0 \frac{\partial}{\partial y}. \tag{6.44}
\]

The other two terms of the \( \chi \)-vector are easily derived in the same way and at leading order, which for \( \chi_2 \) is \( \varepsilon \) and for \( \chi_3 \) is \( \varepsilon^2 \), are

\[
\chi_2 = -\frac{D_0}{Dt} \left( \frac{\partial v_0}{\partial z} \right) = -\frac{D_0}{Dt} \left( \frac{\partial \theta_0}{\partial y} \right), \tag{6.45}
\]

\[
\chi_3 = -\frac{\partial v_0}{\partial z} \frac{D_0}{Dt} \left( \frac{\partial u_0}{\partial z} \right) + \frac{\partial u_0}{\partial z} \frac{D_0}{Dt} \left( \frac{\partial v_0}{\partial z} \right) = \frac{\partial \theta_0}{\partial x} D_0 \frac{\partial \theta_0}{\partial y} - \frac{\partial \theta_0}{\partial y} D_0 \frac{\partial \theta_0}{\partial x}. \]

The leading order result for the stretching rate (6.40) is equivalent to \(- (\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y}) = - \nabla_h \cdot v_1 \), the negative horizontal divergence of the \( O(\varepsilon) \) fields. The baroclinic terms \( (k \times \frac{\partial}{\partial \theta} \nabla \theta) \) has no effect on the form of the leading order stretching rate due to the cancellation of terms due to the thermal wind relations. As these terms cancel, the form that the stretching rate takes is analogous to the stretching rate for the shallow-water equations (c.f. (5.91)). The results for the first two components, \( \chi_1 \) and \( \chi_2 \), of the alignment vector come about because the dominant expressions for these components are simply the first two terms of the \( O(\varepsilon) \) vorticity evolution vector \( \sigma \), in fact, \( \chi_1 \approx -\sigma_2 \) and \( \chi_2 \approx \sigma_1 \). Finally, \( \chi_3 \) is a combination of these two terms along with the \( O(\varepsilon) \) components of the horizontal vorticity, and hence \( \chi_3 \) is at least of order \( \varepsilon^2 \).
6.3 A closer consideration of the evolution equation for the stretching rate

This section is only going to consider the analysis of the evolution equation for the stretching rate but due to the symmetry of all four equations the same comments could equally apply to the evolution equations for $\chi$. By applying the hydrostatic limit ($\delta \to 0$) to equation (6.33) then this equation reduces to

$$\xi_3 \xi \cdot \nabla \left( \theta - \frac{\partial \phi}{\partial z} \right) = 0$$

and as $\xi_3 \neq 0$ the thermal wind relations is retained albeit at one higher derivative. By applying this result to equation (6.33) and substituting in the first few terms for the horizontal components of the absolute vorticity $\xi_1$ and $\xi_2$ then

$$\frac{\xi_2 D\alpha}{\varepsilon} = \xi_2 \left( \chi^2 - \alpha^2 \right)$$

$$+ \left( -\frac{\partial v_0}{\partial z} - \varepsilon \frac{\partial v_1}{\partial z} + \cdots \right) \left\{ \xi \cdot \nabla \left( v - \frac{\partial \phi}{\partial x} \right) - \varepsilon \frac{\partial u}{\partial y} \cdot \nabla \theta - \varepsilon \mathcal{L}u \right\}$$

$$- \left( \frac{\partial u_0}{\partial z} + \varepsilon \frac{\partial u_1}{\partial z} + \cdots \right) \left\{ \xi \cdot \nabla \left( u + \frac{\partial \phi}{\partial y} \right) - \varepsilon \frac{\partial u}{\partial x} \cdot \nabla \theta + \varepsilon \mathcal{L}w \right\} - \xi_3 \xi \mathcal{L}w .$$

(6.46)

The symbols $(1, \varepsilon, \varepsilon^2)$ beneath the under-braces denote the leading order value of each term. Recall that the evolution of the stretching rate and alignment vector is made up primarily of two parts - the equation dependent and independent terms. The equation independent term rises in the natural geometry, that leads to the quaternionic multiplication term $q \otimes q$, of the $(\alpha, \chi)$ variables. The equation independent terms depend on the exact form of the second (material) derivative of the vorticity. Previously the full form of the equations for $\alpha$ and $\chi$ were considered, however, the significance or magnitude of each term was not discussed. By
considering the evolution equations in their non-dimensional form in terms of the Rossby number it is possible to see the relative significance of both terms. Beginning with the full evolution equations in terms of the aspect ratio and Rossby number and then considering the limit $\delta \to 0$, the thermal wind relations are retained. At leading order ($\varepsilon^0$) the geostrophic relations are retained in the $\chi_1$ and $\chi_2$ terms and are apparent in the two other remaining terms. At order $\varepsilon$, the evolution of the stretching rate is solely in terms of the equation dependent terms while the quaternionic geometry of the equation independent term only appears at the higher order of $\varepsilon^2$. This, of course, has not been seen before and although both terms are equally important in the full form of the Lagrangian advection of $(\alpha, \chi)$, at leading order it is the equation dependent term that drive the fluid motion. As a point to note, equation (6.46) and the corresponding forms for $\chi$ are consistent with the results in equations (6.40), (6.43), (6.45) and (6.46) for the leading order forms of the stretching and rotation rates.

6.4 Summary

As our dynamical systems under consideration move further away from the parent dynamics of the Euler equations the equations for the vortex stretching and rotation become even more complicated. This is due to the loss of any mathematical symmetry once approximations are considered in individual, scalar terms in the corresponding momentum equations. For the problem of the hydrostatic, primitive equations the evolution equation for the vortex stretching is quite complex and it is no longer possible to generalise the vortex rotation as a single 3-vector $\chi$. An asymptotic expansion of the vorticity and $(\alpha, \chi)$ variables give the following results at leading order that
\[ \alpha = \varepsilon \frac{\partial w_1}{\partial z} + O(\varepsilon^2), \] (6.47)

\[ \chi_1 = \varepsilon \frac{D_0}{Dt} \left( \frac{\partial u_0}{\partial z} \right) + O(\varepsilon^2), \] (6.48)

\[ \chi_2 = -\varepsilon \frac{D_0}{Dt} \left( \frac{\partial v_0}{\partial z} \right) + O(\varepsilon^2), \] (6.49)

\[ \chi_3 = \varepsilon^2 \left\{ \frac{\partial u_0}{\partial z} \frac{D_0}{Dt} \left( \frac{\partial v_0}{\partial z} \right) - \frac{\partial v_0}{\partial z} \frac{D_0}{Dt} \left( \frac{\partial u_0}{\partial z} \right) \right\} + O(\varepsilon^3). \] (6.50)

Finally a hierarchal structure is evident when particular orders of the full form of the evolution equation for the stretching rate are considered in equation (6.46). Tending the aspect ratio to zero leads to the familiar result of hydrostatic balance being established. At leading order \( O(\varepsilon^0) \) the geostrophic relations are retained. When terms of equivalent order to the Rossby number are considered, the prognostic equation for the leading order stretching rate is governed solely by the equation dependent part of the system. It is only at the higher order of \( \varepsilon^2 \) that the equation independent term, which was the cornerstone of the quaternionic structure, comes into play.
Chapter 7

Conclusions

At the start of this research project there was very little in the way of understanding regarding a quaternionic formulation of three-dimensional fluid flows. In fact, the quaternionic structure in the three-dimensional, incompressible Euler equations had only been discovered the previous year Gibbon (2002). This thesis has developed a new theory regarding the quaternionic structure of three-dimensional flow regimes, of which Euler, is just one particular example. This theory, based on a 4-vector $q$ comprising the vortex growth and rotation rates, says that the evolution of the 4-vector $q$ satisfies

$$
\frac{Dq}{Dt} + q \otimes q + \frac{1}{\mathbf{w} \cdot \mathbf{w}} \frac{D^2 \mathbf{w}}{Dt^2} \otimes \mathbf{w} = 0, \quad (7.1)
$$

where $\mathbf{w}$ is a 4-vector representation of the vorticity satisfying

$$
\frac{D\mathbf{w}}{Dt} = q \otimes \mathbf{w}. \quad (7.2)
$$

This general equation (7.1) consists of two parts - the equation dependent and independent terms. The equation independent term $(q \otimes q)$ is due to the suitable geometry of the quaternionic representation of the vorticity variables while the equation dependent term is constantly changing as a hierarchy of different flow regimes are considered. A number of key results with slight variations have been used throughout this thesis, for example, Ertel’s theorem which has been used widely to calculate
the explicit form of the vortex stretching vector has also been the crux for the “closing” of the Lagrangian advection equation for $q$. A second equally important point is the role of the constraint equation. This transforming of vorticity variables increases the number of prognostic equations by one and so a constraint is vital to provide information regarding the dependent variables within the flow. This constraint equation, although not considered in any particular detail in this research problem, plays a vital role in our understanding of how the dependent variables interact within the flow and could also be the link between the quaternionic structure in this research and that of the quaternionic structure in the balanced models, as the constraint equation leads to a particular type of Monge-Ampere equation which is at the heart of the quaternionic structure of the balanced models.

This research has tried to tackle a hierarchy of different fluid flows from the Euler equations with rotation to the hydrostatic, primitive equations. Numerous problems have occurred with certain systems and these have mostly been overcome, such as the breakdown of the quaternionic structure to a single scalar equation for the stretching rate when there is perfect alignment between the vorticity and vorticity stretching vectors to the need to non-dimensionalise the equations of motion to successfully apply the case of hydrostatic balance. However, as more and more approximations are made, the equations governing the evolution of the stretching rate and alignment vector get more involved, and my belief is that there is no real advantage in trying to derive the full form of these equations in even more complicated vorticity representations such as would be the case for say the semi-geostrophic equations.

However, the approach of considering the leading order terms to get an understanding of these variables may be useful and could open the door for the application of these vorticity variables in the practical sense of weather forecasting or in a suitable diagnostic context. One further advantage of dealing specifically with the non-dimensional form of the equations of motion is that it becomes quite clear what the relative orders of each term should be and hence the particular dominant feature of each term. This of course is part of the larger framework of the governing
equations dependent and independent terms. It was not at all obvious that it would be the equation dependent terms that in effect would drive the motion at leading order for the hydrostatic, primitive equations.

The last word should go to quaternions as they are at the heart of this research problem. After quaternions went out of favour in the last century or so, they are now being used in such diverse areas as computer graphics to orbital mechanics and control theory. They continue to play an important role in the study of manifolds and it is believed that a natural quaternionic structure will point to a similar geometric structure in these partial differential equations that govern the flow of fluids. Although we are still a long way from understanding these fluid flow equations, this thesis has hopefully given an insight into the strength of quaternions in providing an algebraic framework for the study of a wide range of geophysical fluid systems.
Chapter 8

Appendix - numerical treatment of the vortex stretching and rotation variables

The aim of this chapter is to consider the representation of the stretching rate and alignment vector using data from the Meteorological Office’s Unified Model (UM). The corresponding equations must first be transformed from a Cartesian set of coordinates \((x, y, z)\) to spherical polar co-ordinates \((\lambda, \phi, r)\). The grid structure is then discussed and key quantities are discretized. These are then plotted and discussed in light of previous derived theoretical results. The second part of this chapter considers numerically the earlier restriction for the development of singular solutions, namely that the matrix \(P' + \Omega^*\), controls the development of singular solutions and is in fact an upper bound for the growth of the vortex stretching rate and ultimately the vorticity.

8.1 The momentum and vorticity equation in spherical polar co-ordinates

Recall that the momentum equation for an inviscid fluid flow is given by
Figure 8.1: The unit vectors $\mathbf{I}, \mathbf{J}, \mathbf{K}$ associated with the directions $Ox, Oy, Oz$ in the rotated system and the unit vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$ associated with the zonal, meridional and radial directions at a point $P$ having longitude $\lambda$ and latitude $\phi$ in the related system.

The relationship between these two co-ordinate systems is given by

$$x = r \cos \phi \cos \lambda,$$
$$y = r \cos \phi \sin \lambda,$$
$$z = r \sin \phi,$$  

(8.2)
\[ \delta x = h_1 \delta \lambda e_\lambda + h_2 \delta \phi e_\phi + h_3 \delta r e_r \]  
(8.3)

where the metric factors \( h_i \) are defined by

\[ h_i^2 = \sum_{j=1}^{3} \left( \frac{\partial x_j}{\partial q_i} \right)^2, \]  
(8.4)

where the \( q_i \) are the spherical polar co-ords and the \( e_\lambda, e_\phi, e_r \) denote unit vectors parallel to the co-ordinate lines and in the direction of increase in these co-ordinates. For spherical-polar co-ordinates the metric factors are

\[ h_1 = r \cos \phi, \quad h_2 = r, \quad h_3 = 1, \]  
(8.5)

and

\[ e_\lambda = -\sin \lambda e_1 + \cos \lambda e_2, \]
\[ e_\phi = -\sin \phi \cos \lambda e_1 - \sin \phi \sin \lambda e_2 + \cos \phi e_3, \]  
(8.6)
\[ e_r = \cos \phi \cos \lambda e_1 + \cos \phi \sin \lambda e_2 + \sin \phi e_3, \]

where \( e_{1,2,3} \) correspond to unit vectors in the \((x, y, z)\)-plane respectively. Denoting the zonal, meridional and radial velocity components as \( u_\lambda, u_\phi, u_r \) respectively the advection term in equation (8.1) is given by the expansion of the following expression

\[ \left[ \frac{u_\lambda}{r \cos \phi} \frac{\partial}{\partial \lambda} + \frac{u_\phi}{r} \frac{\partial}{\partial \phi} + u_r \frac{\partial}{\partial r} \right] (u_r e_r + u_\phi e_\phi + u_\lambda e_\lambda). \]  
(8.7)

See glossary for the specific form that the gradient, divergence and curl take in a spherical-polar co-ordinate system. This term can be expanded using the expressions given for the unit vectors in equation (8.6) and on expanding the Coriolis term in equation (8.1) the individual components of the momentum equation (8.1) in spherical-polars is given by
\[
\frac{Du_\lambda}{Dt} + \frac{u_\lambda u_r}{r} - \frac{u_\lambda u_\phi}{r} \tan \phi - 2\Omega u_\phi \sin \phi + 2\Omega u_r \cos \phi = - \frac{1}{\rho r \cos \phi} \frac{\partial p}{\partial \lambda}, \tag{8.8}
\]

\[
\frac{Du_\phi}{Dt} + \frac{u_\phi u_r}{r} + \frac{u_\phi^2}{r} \tan \phi + 2\Omega u_\lambda \sin \phi = - \frac{1}{\rho r} \frac{\partial p}{\partial \phi}, \tag{8.9}
\]

\[
\frac{Du_r}{Dt} - \frac{u_\lambda^2}{r} - \frac{u_\phi^2}{r} - 2\Omega u_\lambda \cos \phi = - \frac{1}{\rho} \frac{\partial p}{\partial r} - g. \tag{8.10}
\]

the material derivative is

\[
\frac{D}{Dt} = \frac{\partial}{\partial t} + \frac{u_\lambda}{r \cos \phi} \frac{\partial}{\partial \lambda} + \frac{u_\phi}{r} \frac{\partial}{\partial \phi} + u_r \frac{\partial}{\partial r}. \tag{8.11}
\]

The absolute vorticity \( W = \nabla \times u + 2\Omega \) can be calculated by considering the curl in spherical-polar co-ordinates and in component form \((\omega_\lambda, \omega_\phi, \omega_r)\) is given by

\[
\omega_\lambda = \frac{1}{r} \left\{ \frac{\partial u_r}{\partial \phi} - \frac{\partial}{\partial r} (ru_\phi) \right\}, \tag{8.12}
\]

\[
\omega_\phi = \frac{1}{r} \frac{\partial}{\partial r} (ru_\lambda) - \frac{1}{r \cos \phi} \frac{\partial u_r}{\partial \lambda} + 2\Omega \cos \phi, \tag{8.13}
\]

\[
\omega_r = \frac{1}{r \cos \phi} \left\{ \frac{\partial u_\phi}{\partial \lambda} - \frac{\partial}{\partial \phi} (u_\lambda \cos \phi) \right\} + 2\Omega \sin \phi. \tag{8.14}
\]

For convenience, in later sections, the spherical-polar velocity variables \((u_\lambda, u_\phi, u_r)\) will be replaced by \((u, v, w)\). For a detailed discussion of the momentum equations and corresponding vorticity equations, derived in spherical polar coordinates, in this chapter, see White (2002) and White et al. (2005).

### 8.2 The grid structure

#### 8.2.1 The co-ordinate system

The equations of motion and components of vorticity have been formulated in terms of the three independent, spherical-polar co-ordinates \((\lambda, \phi, r)\) given in equations (8.8)-(8.10) and (8.12) - (8.14) respectively. In terms of these variables, the approximation to mean sea level is given by \(r = a\) where \(a\) is the mean radius of the
Chapter 8. Appendix - numerical treatment of the vortex stretching and rotation variables

Earth. The vertical component is transformed into a generalised “terrain-following” vertical component \( \eta \), and

\[
\eta = \eta (r, r_S, r_T), \tag{8.15}
\]

where \( \eta = 0 \) on \( r = r_S(\lambda, \phi) \) - height of the Earth’s local surface that deviates from the mean sea level value, due to only orographic features and \( \eta = 1 \) on \( r = r_T \)
where \( r_T \) is the top of the model domain and is constant. Therefore the vertical co-ordinate is

\[
r = r(\lambda, \phi, \eta), \tag{8.16}
\]

### 8.2.2 Grid Spacing and variable placement

The continuous equations, defined in (8.8)-(8.10) and (8.12) - (8.14) need to be discretized on grids that need to be independent of each of the three model co-ordinates \((\lambda, \phi, \eta)\). As each grid is independent of the others, any point on this discrete mesh of grid points can be expressed by the three indices \((i, j, k)\). These indices each identifies a particular co-ordinate plane so therefore in each place one of the three co-ordinates is held constant. These three planes are \( \phi - \eta \) for the \( i \) indice, \( \lambda - \eta \) for \( j \) and \( \lambda - \phi \) for \( k \). To ease explicit calculations of such quantities like for example the vorticity, the grids are staggered in all three directions. In the horizontal (the \( \lambda - \phi \)) an Arakawa C-grid is used whilst in the vertical (the \( \phi - \eta \) and \( \lambda - \eta \) planes) the Charney-Philips grid staggering is used.

In each of the three co-ordinate planes there are two distinct grid structure, and in fact each alternates with the next. To help distinguish the grid type, each index is assigned either an integral or half-integral value. For example, \( i \) has an integral value \( I \), or half-value, \( I \pm 1/2 \), and so on. The data for \( u \) is stored at \( (I, J \pm 1/2, K \pm 1/2) \), \( v \) at \( (I \pm 1/2, J, K \pm 1/2) \) and \( w \) at \( (I \pm 1/2, J \pm 1/2, K) \). The vertical integral \( K \) and half-integral values \( K \pm 1/2 \) are labelled as \( \theta \) and \( \rho \)-levels respectively as these are the particular height levels where these model
variables are stored (where $\theta$ represents all variations of thermodynamic variables).

Figures 8.2 and 8.3 show the grid-staggering and vertical structure of the Arakawa C-grid and Charney-Philips respectively.

### 8.3 Discretization of model variables

Recall that the vorticity component in the radial direction was given by the expression

$$\omega_r = \frac{1}{r \cos \phi} \left\{ \frac{\partial u_\phi}{\partial \lambda} - \frac{\partial}{\partial \phi} \left( u_\lambda \cos \phi \right) \right\} + 2\Omega \sin \phi.$$  \hfill (8.17)

So for a particular height level, the variables in this equation must be discretized in terms of the grid-points $(i, j)$, so for example
Figure 8.3: Charney-Philips grid staggering. The $\theta$ and $\rho$-levels correspond to the integral value $K$ and half-integral values $K \pm 1/2$ respectively. The height of $\eta$ level is shown as the sum of the three parts, $r(E)$ - the mean radius of the Earth, $r(O)$ - the height due to orography and $r(\rho, \theta)$ the height at a particular $(\rho, \theta)$ level

$$u \cos \phi = u(i, j) \cos \phi_u(i, j),$$

where $\phi_u(i, j)$ is the latitude at the corresponding point $u(i, j)$. If the $j$ index has a range of values from 1 to $n$ then $\phi_u(i, 1) = -\pi/2$ (south-pole) and $\phi_u(i, n) = \pi/2$ (north-pole) and so

$$\phi_u(j) = \frac{(2j - (n + 1))\pi}{2(n - 1)}, \quad (8.18)$$

there is a slightly different expression for $\phi_v(i, j)$ because there is one less grid point in the $j$-index for the variable $v$ as the poles are staggered with the $u$-levels. The full discretization of the continuous equation (8.17) is given by

$$2\Omega \cos \phi_v(j) + \frac{1}{r(i, j) \cos \phi_v(j)} \left( \frac{v(i + 1, j) - v(i, j)}{\Delta \lambda} - \frac{1}{\Delta \phi} \left\{ u(i, j + 1) \cos \phi_u(j + 1) - u(i, j) \cos \phi_u(j) \right\} \right), \quad (8.19)$$

where $r(i, j)$ is the height at the prescribed vertical level, and $\Delta \lambda$ and $\Delta \phi$ are the space between grid-points in the longitude and latitude directions respectively. All other variables are discretized in a similar way.
8.4 UM model data and grid spacing

Numerical data for all three components of the velocity field, the pressure and density fields at all grid points and height levels was obtained from the UK Meteorological Office’s Unified Model (UM). The data was from the 01/12/2003 at 12z and was obtained at a grid spacing of approximately 10km at mid-latitudes. All numerical calculations were considered on the vertical $\theta$-level two - approximately 0.1km above the orography.

8.5 Numerical consideration of the different vortex variables

8.5.1 Numerical representation of the vorticity components

In equation (8.19), the radial component of vorticity $\omega_r$ was discretized. Considering the same form for the other two components of vorticity, albeit having to consider the change over vertical grid points, the three components of vorticity are plotted along with their maximum, minimum and mean values in Figures 8.4 - 8.6. The colour scaling of the $\omega_\lambda$ and $\omega_\phi$ components are the same, because they are equal in magnitude, but the $\omega_r$ colour scaling is of order one magnitude less.
Chapter 8. Appendix - numerical treatment of the vortex stretching and rotation variables

8.5.2 The vortex stretching components

To calculate the corresponding components of vortex stretching, the evolution equation for each component of the vorticity has to be derived. They take the explicit
Chapter 8. Appendix - numerical treatment of the vortex stretching and rotation variables

Figure 8.6: The third component of the vorticity vector $\omega_r$

![Image of a map with color gradient indicating vorticity values]

the corresponding maximum, minimum and mean values for the vortex stretching together with the plots of these fields can be seen in Figures 8.7 - 8.9.

8.5.3 The stretching rate and negative horizontal divergence

With the numerical analysis of the vorticity and vortex stretching complete, it is possible to combine the two expressions for the purposes of modelling the vortex stretching rate. Also, recall that at leading order the stretching rate was given by the negative horizontal divergence of the velocity field, which in spherical polars is given by

\[
\frac{D\omega_\lambda}{Dt} = (\omega \cdot \nabla) u + \frac{1}{r} (\omega_\lambda w - \omega_\lambda v \tan \phi + u \omega_\phi \tan \phi - u \omega_r), \quad (8.20)
\]

\[
\frac{D\omega_\phi}{Dt} = (\omega \cdot \nabla) v + \frac{1}{r} (\omega_\phi w - \omega_r v), \quad (8.21)
\]

\[
\frac{D\omega_r}{Dt} = (\omega \cdot \nabla) w, \quad (8.22)
\]
Chapter 8. Appendix - numerical treatment of the vortex stretching and rotation variables

121

Figure 8.7: The first component of the vortex stretching term $\sigma_{\lambda}$

Figure 8.8: The second component of the vortex stretching term $\sigma_{\phi}$

\[-\nabla \cdot \mathbf{v} = -\frac{1}{r \cos \phi} \left( \frac{\partial u}{\partial \lambda} + \frac{\partial}{\partial \phi} (v \cos \phi) \right).\] (8.23)
The stretching rate is given in Figure 8.10 while the negative divergence rate is shown in Figure 8.11. Comparing the two plots and observing the order of magnitude of the mean values for each then numerically the horizontal divergence is a good leading order estimate for the vortex stretching rate.

### 8.5.4 The components of the vortex alignment variable

The three components of the vortex alignment vector are in essence the cross product of the vorticity with the vorticity stretching and so the three components are given in Figures 8.12 - 8.14.

### 8.6 The numerical analysis of the development of singular solutions

From Chapter 3.9, the result was derived that the maximum vortex stretching rate is bounded by the maximum row sum of the pressure Hessian matrix $P$. With the
Chapter 8. Appendix - numerical treatment of the vortex stretching and rotation variables

Figure 8.10: The stretching rate $\alpha$

Figure 8.11: The negative horizontal divergence field $-\nabla \cdot \mathbf{v}$

added effect of rotation $P$ became $P' + \Omega^*$ where $P'$ has an additional set of terms representing gravity (see section 4.3) and $\Omega^*$ is given in (4.43). Also recall that the
Chapter 8. Appendix - numerical treatment of the vortex stretching and rotation variables

Figure 8.12: The first component of the vortex rotation vector $\chi_\lambda$

Figure 8.13: The second component of the vortex rotation vector $\chi_\phi$

stretching rate is bounded by the variable $X$ where $X^2 = \alpha^2 + \chi \cdot \chi$. Numerically, the $X$-variable is shown in figure 8.15 and the maximum row sum of the matrix
Figure 8.14: The third component of the vortex rotation vector $\chi_r$

\[ X \text{ field} \]

Figure 8.15: The $X$ variable given by $X^2 = \alpha^2 + \chi \cdot \chi$

$P' + \Omega^*$ is given in figure 8.16.

For all positive values of the stretching rate the maximum row sum of the ma-
Figure 8.16: The maximum row sum of the matrix \((P' + \Omega^*)\)

The maximum row sum of the matrix \((P' + \Omega^*)\) is greater than the corresponding values for the stretching rate and the \(X\) variable, defined as \(X = \alpha^2 + \chi \cdot \chi\).

### 8.7 Summary

In this chapter the vorticity and vortex stretching vector have been defined in terms of spherical, polar co-ordinates. On suitably staggered grids these variables have been modelled using data obtained from the Met Office UM. Very little, if anything, has been said about the particular structure that these variables take numerically. The thinking behind this chapter was to give the reader an insight into the visual and numerical forms of these variables and in no way has it been suggested that these variables could be used as a possible set of diagnostic variables for applications in forecasting or numerical weather prediction. It is, however, encouraging that the stretching rate at leading order is numerically very similar to the negative horizontal divergence which was one of the key theoretical results of Chapters 5 and 6. In the final part of this chapter the numerical data was used to show that the Hessian
matrix \((P' + \Omega^*)\) is indeed a bound on the stretching rate and is therefore another condition on the development of any singular solutions in the corresponding partial differential equations.
Chapter 9

Glossary

9.1 Glossary of mathematical symbols

9.1.1 Chapter 2

\(\frac{D}{Dt}\) - the material derivative operator in three-dimensional space

\(\{\omega, |\omega|, \hat{\omega}\}\) - general (unspecified) vorticity, vorticity magnitude, vorticity unit vector

\(\sigma\) - corresponding vorticity stretching vector

\(\alpha\) - vorticity stretching rate/rate of change of vorticity magnitude

\(\{\chi, \chi, \hat{\chi}\}\) - vorticity alignment vector/rate of change of vorticity direction, corresponding scalar magnitude, unit vorticity alignment vector

\(\phi\) - local angle between the vectors \(\omega\) and \(\sigma\)

\(\{i, j, k\}\) - hypercomplex numbers

\(I\) - identity matrix

\(\{e_i, \sigma_i\}\) - bases matrices/Pauli matrices

\(\{1, i, j, k\}\) - unit 4-vectors isomorphic to \(\{I, e_i\}\)

\(\{q, s_1, s_2, w\}\) - 4-vectors based on the \(\{\alpha, \chi, \sigma, \omega\}\) variables

\(\otimes\) - direct product of two 4-vectors

\(\{\zeta_e, \psi_e, \gamma_e\}\) - complex variables based on the \(\{\alpha, \chi, \sigma, \omega\}\) variables
9.1.2 Chapter 3

\( \mathbf{u}(x, t) \) - velocity field

\( p(x, t) \) - pressure field

\( \rho(x, t) \) - density field

\( \mathbf{\omega} = \nabla \times \mathbf{u} \) - vorticity field

\( S = \frac{1}{2} (u_{ij} + u_{ji}) \) - strain matrix

\( ||\mathbf{\omega}(\cdot, t)||_{m,\infty} \) - \( L^m,\infty \)- norms respectively

\( P = \{p, ij\} \) - pressure Hessian matrix

\( \{\alpha_p, \chi_p\} \) - corresponding \( \{\alpha, \chi\} \) variables with the strain matrix replaced by the pressure Hessian matrix

\( \mathbf{q}_p \) - corresponding 4-vector representation of \( \{\alpha_p, \chi_p\} \)

\( \Delta = \partial_{ii} \) - 3-D Laplacian operator

\( \alpha \) - particle labels

\( U \) - potential based on the complex variable \( \psi_c \)

\( \gamma_k, \tau_k \) - eigenfunctions for the solutions to the Schrodinger problem and corresponding arbitrary (complex) constants

\( \{X, X_p\} \) - co-ordinate transformations of the \( \{\alpha, \chi\} \) & \( \{\alpha_p, \chi_p\} \) variables

\( \{\mathbf{u}, \mathbf{p}, \nabla\} \) - 4-vector representation of \( \{\mathbf{u}, p, \nabla\} \) respectively

\( \nu \) - viscosity

\( \{\mu, \lambda, \epsilon\} \) - unknown scalars of the pressure Hessian analysis

\( \mathbf{q}_{\mu, \lambda, \epsilon} \) - corresponding 4-vector for the evolution of \( D\mathbf{q}_p/Dt \)

9.1.3 Chapter 4

\( \phi(x) \) - external potential

\( \Omega \) - angular velocity

\( \xi \) - absolute vorticity

\( \mathbf{q}_\mu \) - modified pressure Hessian variables

\( \mathbf{q}_{\Omega} \) - 4-vector representation of the angular velocity vector

\( \rho \) - density 4-vector
Chapter 9. Glossary

9.1.4 Chapter 5

\( \zeta \) - vertical component of the absolute vorticity
\( \xi \) - vertical component of the relative vorticity
\( \mu \) - diffusion coefficient
\( f \) - forcing term
\( \nabla_\perp \) - curl operator
\( \eta \) - \( \nabla_\perp \theta / |\nabla_\perp \theta| \) where \( \theta \) is the potential temperature

9.1.5 Chapter 6

\( \phi \) - geopotential
\( \theta \) - potential temperature
\( \theta_r \) - constant reference potential temperature

9.1.6 Chapter 7

(\( \lambda, \phi, r \)) - spherical, polar co-ordinates
\( h_i \) - metric factors
(\( \omega_\lambda, \omega_\phi, \omega_r \)) - vorticity components
\( \eta \) - surface levels
(\( \Delta \lambda, \Delta \phi \)) - grid spacing in the longitudinal and latitudinal directions

9.2 Vector and scalar laws

Given vector fields \( \mathbf{F}, \mathbf{G} \) and scalar field \( \Phi \) then the following laws for operations with the operator \( \nabla \) hold
\[ \nabla \times (F \times G) = (G \cdot \nabla) F - (F \cdot \nabla) G + F (\nabla \cdot G) - G (\nabla \cdot F), \]
\[ \nabla (F \cdot G) = (F \cdot \nabla) G + (G \cdot \nabla) F + F \times (\nabla \times G) + G \times (\nabla \times F), \]
\[ \nabla \times (\nabla \times F) = \nabla (\nabla \cdot F) - \nabla^2 F, \]
\[ \nabla \cdot (\Phi F) = \Phi \nabla \cdot F + (\nabla \Phi) \cdot F, \]
\[ \nabla \cdot (\nabla \times F) = 0, \]
\[ \nabla \times (\nabla \Phi) = 0. \]

### 9.3 Integral theorems

Functions are assumed to be continuously differentiable.

Gauss’ Theorem or Divergence Theorem:

\[ \int_V \nabla \cdot F \, dV = \int_S F \cdot \hat{n} \, dS, \]

where \( S = \partial V \) closed surface.

Stokes’ theorem:

\[ \int_S (\nabla \times F) \cdot dS = \oint_C F \cdot dr, \]

where \( C = \partial S \) closed curve.

### 9.4 Spherical-polar form of vector operators

The vector gradient operator is given by

\[ \nabla \equiv \left( \frac{1}{r \cos \lambda} \frac{\partial}{\partial \lambda}, \frac{1}{r} \frac{\partial}{\partial \phi}, \frac{\partial}{\partial r} \right). \tag{9.1} \]

The divergence of a vector field \( A = (A_\lambda, A_\phi, A_r) \) is given by

\[ \nabla \cdot A \equiv \frac{1}{r \cos \phi} \left( \frac{\partial A_\lambda}{\partial \lambda} + \frac{\partial}{\partial \phi} (A_\phi \cos \phi) \right) + \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 A_r \right). \tag{9.2} \]
The curl of \( \mathbf{A} \) is given in component form by

\[
(\nabla \times \mathbf{A})_\lambda \equiv \frac{1}{r} \left( \frac{\partial A_r}{\partial \phi} - \frac{\partial}{\partial r} (rA_\phi) \right),
\]

(9.3)

\[
(\nabla \times \mathbf{A})_\phi \equiv \frac{1}{r} \left( \frac{\partial}{\partial r} (rA_\lambda) - \frac{1}{\cos \phi} \frac{\partial A_r}{\partial \lambda} \right),
\]

(9.4)

\[
(\nabla \times \mathbf{A})_r \equiv \frac{1}{r \cos \phi} \left( \frac{\partial A_\phi}{\partial \lambda} - \frac{\partial}{\partial \phi} (A_\lambda \cos \phi) \right).
\]

(9.5)
References


Gibbon, J., Galanti, B., and Kerr, R. (2000). *Turbulence structure and vortex dynamics*, chapter Stretching and compression of vorticity in the 3D Euler equa-


References


