An axisymmetric analytic model of baroclinic instability

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1. Introduction

Baroclinic instability is the dominant mechanism by which synoptic-scale cyclones and anti-cyclones are generated in the mid-latitudes. Here we present a new simple mathematical model of the instability process which we are using to better understand the nonlinear behaviour of breaking baroclinic waves.

2. Mathematical Background

The two archetypal analytic models of baroclinic instability are the Eady and Charney models. The model presented here is most similar to the Eady model in that it uses quasi-geostrophic dynamics with a uniform distribution of potential vorticity. This focuses attention on the important roll of the temperature distributions on the lower boundary and the tropopause. As in the Eady model, both of these are taken to be flat horizontal boundaries.

The mathematical formulation is based on the conservation of the quasi-geostrophic potential vorticity anomaly throughout the atmosphere (Vallis, 2006):

$$\frac{Dq}{Dt}=0, \qquad q=\nabla_h^{2}\psi+\frac{f^2}{N^2}\frac{\partial^2\psi}{\partial z^2} \tag{1}$$

where ψ is the streamfunction, f the (constant) Coriolis parameter and N the Brunt-Vaisala buoyancy frequency. Together with the conservation of potential temperature on the upper and lower boundaries:

$$\frac{D\theta}{Dt}=0, \qquad \theta=\frac{\partial\psi}{\partial z} \tag{2}$$

these equations uniquely specify the streamfunction.

3. Model Setup

The setup we study here is sketched in Figure 1. It takes the form of two vertically aligned boundary temperature anomalies, one at the surface and one at the tropopause level. Each is a circular patch of uniform temperature.

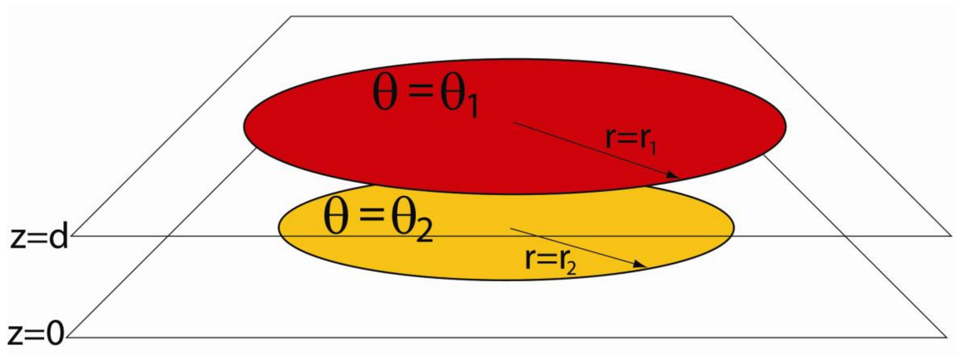


Figure 1: The model setup. The atmosphere is bounded by rigid horizontal boundaries at heights z=0 and d. Each boundary contains a circular patch of anomalous temperature.

The major differences with the Eady model are (1) that the temperature gradients on the boundaries are concentrated into sharp fronts, and (2) that circular geometry is used which allows a much wider range of basic states to be studied, as well as potentially exhibiting more realistic wave breaking behaviours.

Analytic progress is possible because of the models simplicity: the dynamics are governed entirely by the positions of the two patch edges.

There are three parameters in this model: the ratio of the upper and lower temperature values, $\nu=\theta_2/\theta_1$, the ratio of the patch radii $\lambda=r_2/r_1$ and d the depth of the troposphere.

Cross sections of the basic state zonal wind and potential temperature are shown in Figure 2 for two example sets of parameter values. Note that the regions of strong meridional temperature gradient, and the jets, are localised around the fronts, a feature absent from both the Eady and Charney models.

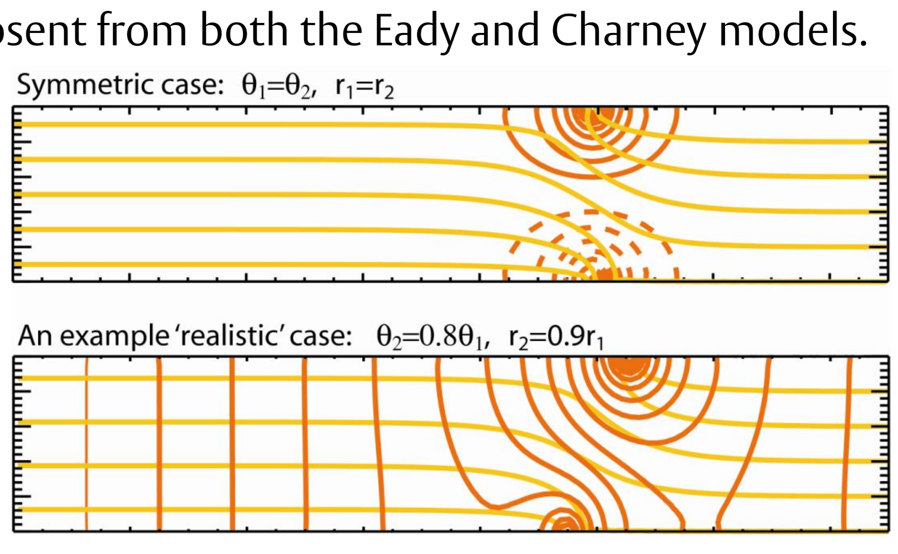


Figure 2: Example cross sections of the basic state zonal wind (orange) and potential temperature (yellow) profiles for two example sets of parameter values.

3. Linear analysis

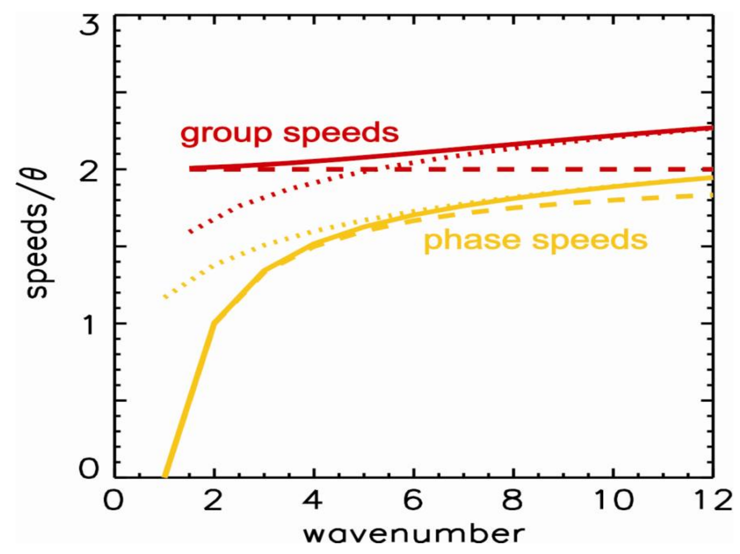
For small amplitude disturbances of the patch edges, the evolution of the system is given by a simple 2x2 matrix equation:

$$i\frac{d}{dt}\begin{pmatrix}\eta_1\\ \eta_2\end{pmatrix}=\begin{pmatrix}P_1 & I_{12}\\ -I_{21} & -P_2\end{pmatrix}\begin{pmatrix}\eta_1\\ \eta_2\end{pmatrix} \tag{3}$$

where $r_i+\eta_i(t)e^{in\varphi}$ are the positions of the patch edges for a wave number n disturbance. The functions P_i represent the wave propagation of each edge wave on its own temperature front and I_{ij} the interaction between the two patches.

These functions can be calculated quasi-analytically in terms of Bessel functions. There is some subtlety in deriving the propagation coefficients since sharp QG temperature fronts are associated with velocity singularities (see Figure 2). However, the problem is solved in Harvey & Ambaum (2010). The result is shown in Figure 3 as a function of the wave number n. There are interesting comparisons between these phase speeds and both those of the Rankine vortex in 2-d Euler dynamics and also the corresponding surface QG result (Harvey & Ambaum 2010).

Figure 3: The phase and group speeds of disturbances on a single patch. The phase speeds enter (3) as the P_i terms. The dashed and dotted lines show the corresponding results for the 2-d Euler Rankine vortex and the pure surface QG cases respectively.



There are normal mode solutions to (3), with the normal mode growth rates given by the matrix determinant: $\sigma=\sqrt{I_{12}I_{21}-P_1P_2}$. Figure 4 shows the normal mode growth rates for the two example sets of parameter values used in Figure 2.

To illuminate some features of the instability further, Figure 5 shows the regions of parameter space with unstable normal modes.

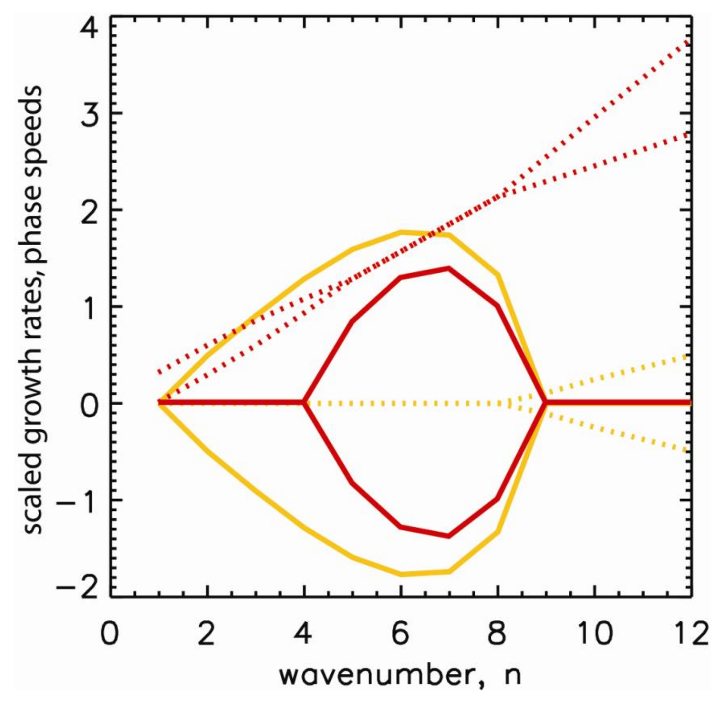


Figure 4: The normal mode growth rates (solid) and phase speeds (dashed) for the two example cases of Figure 2.

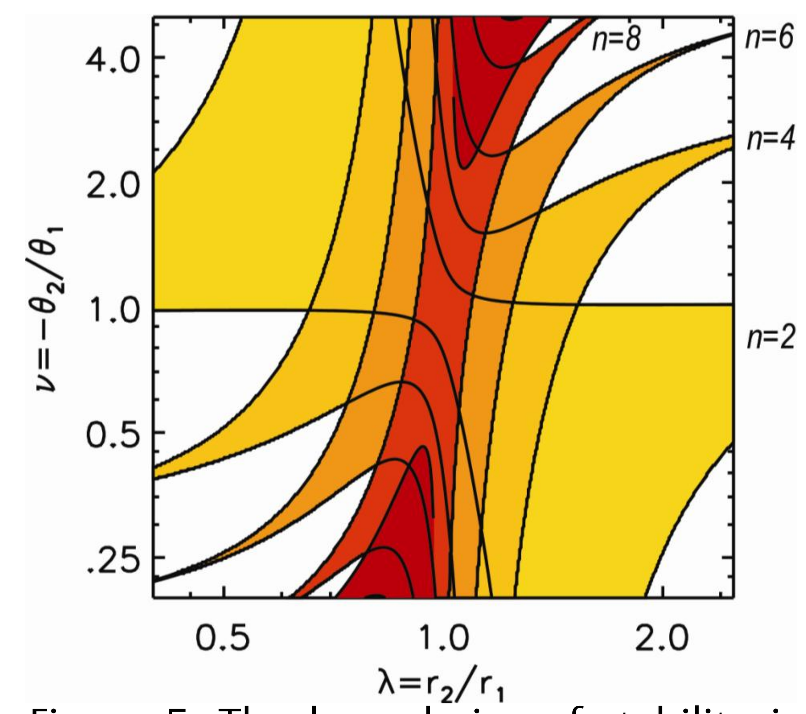


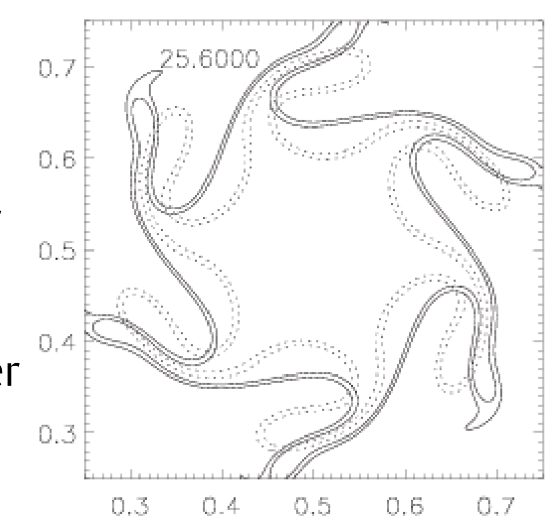
Figure 5: The boundaries of stability in parameter space. The two example cases are indicated by the letters.

5. Conclusions

We have analysed the linear stability properties of a new model for baroclinic instability.

We plan to numerically integrate the full nonlinear equations in order to study the wave breaking behaviour of this setup.

The figure shows an example integration using parameter values from the second example case above.



This case appears to break anti-cyclonically, but others behaviours are seen as the parameters are varied.

References

Harvey, B. J. and Ambaum, M. H. P., Perturbed Rankine vortices in surface quasi-geostrophic dynamics *Geophys. & Astro. Fluid Dyn.* (2010), doi:10.1080/03091921003694719

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