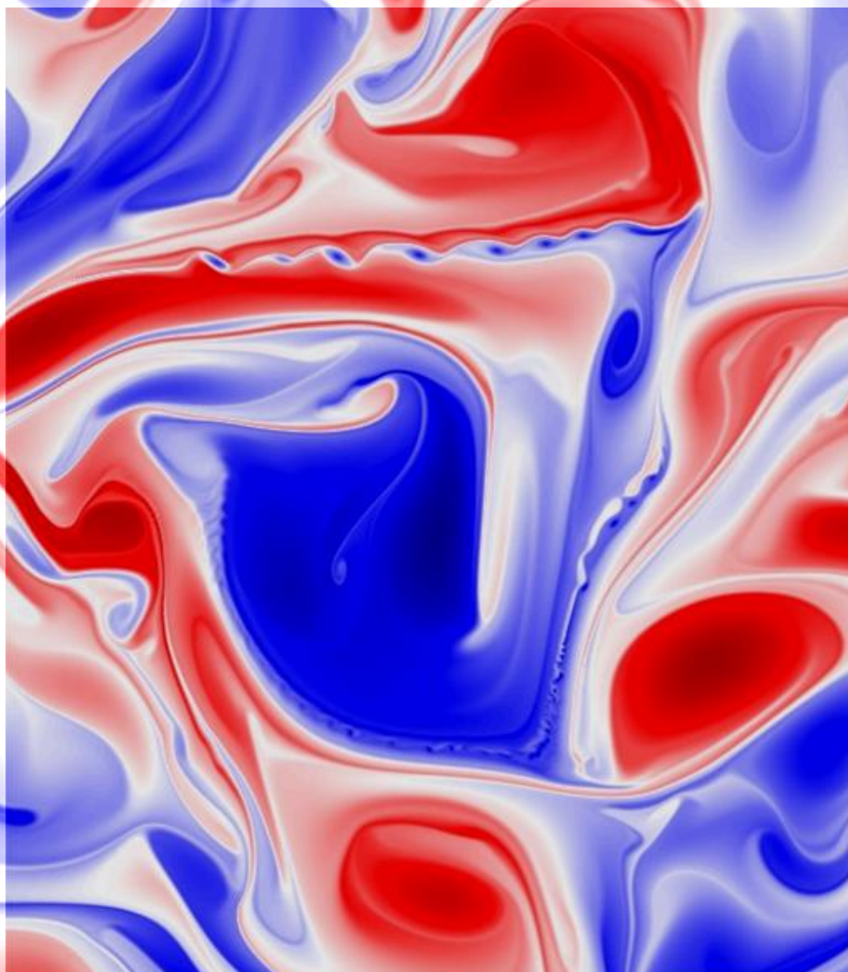
Filament instability in surface quasi-geostrophic turbulence



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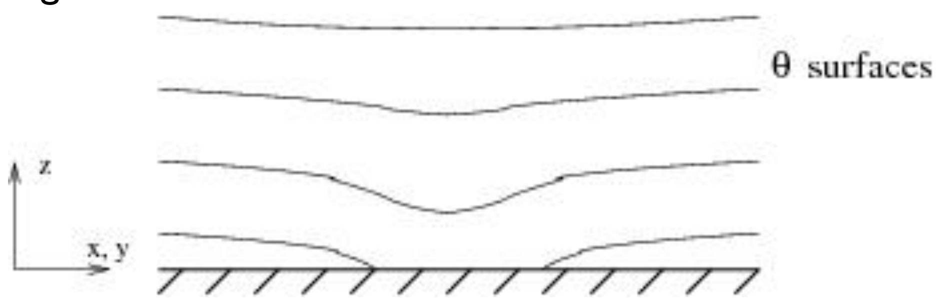
What is surface quasi-geostrophic (SQG) dynamics?

•Models a rotating stratified fluid near a horizontal boundary

•Consider quasi-geostrophic motion with uniform interior potential vorticity:

E.g.



QGPV anomaly is zero: $q = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{f^2}{N^2}\frac{\partial^2}{\partial z^2} \right) \psi_g = 0$ in z>0,

with potential temperature $\theta = \frac{\partial \psi_g}{\partial z}$ conserved at the boundary: $\frac{D\theta}{Dt} = 0$ at z=0.

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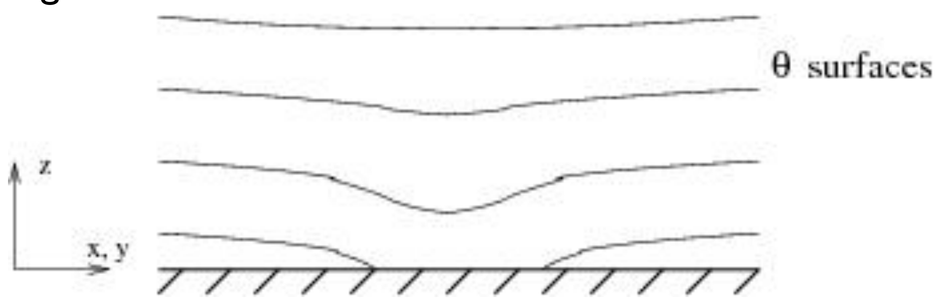

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This is a 2-d advection equation with inversion given by

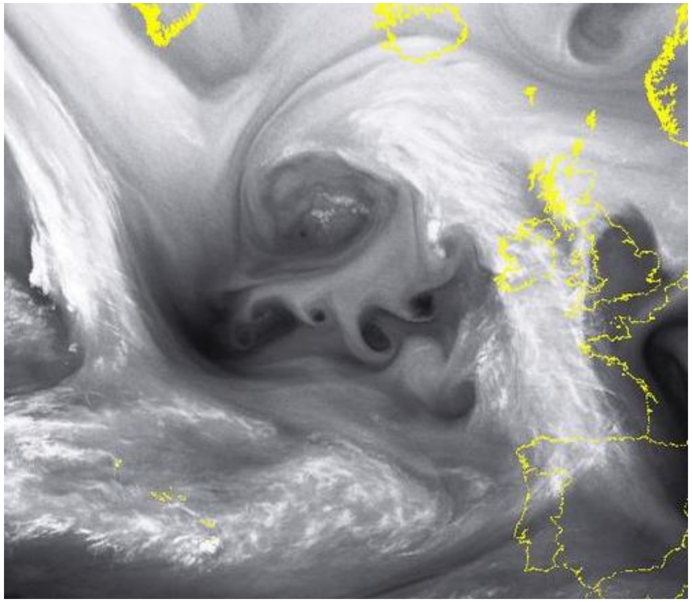
$$\hat{\psi}_g(\mathbf{k}) = -\frac{\hat{\theta}(\mathbf{k})}{|\mathbf{k}|} \quad \text{or} \quad \psi_g(\mathbf{x}) = -\frac{1}{2\pi} \int\int \frac{\theta(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} d^2\mathbf{x}'$$

Applications

There are several…

- Surface frontal dynamics (e.g. Schär and Davies (1990)) – SQG is the QG version of the semigeostrophic equations.

- Tropopause dynamics (Juckes (1994)) SQG describes perturbations to the tropopause:



- Upper level ocean dynamics (Lapeyre and Klein 2006)

Comparison to Euler equations - I

2-d Euler equations

$$\frac{D\xi}{Dt} = 0 \qquad \hat{\psi}(\mathbf{k}) = -\frac{\hat{\xi}(\mathbf{k})}{|\mathbf{k}|^2}$$

Conserved quantities:

$$E = \int \psi\xi dA$$

$$Z = \int \xi^2 dA$$

SQG equations

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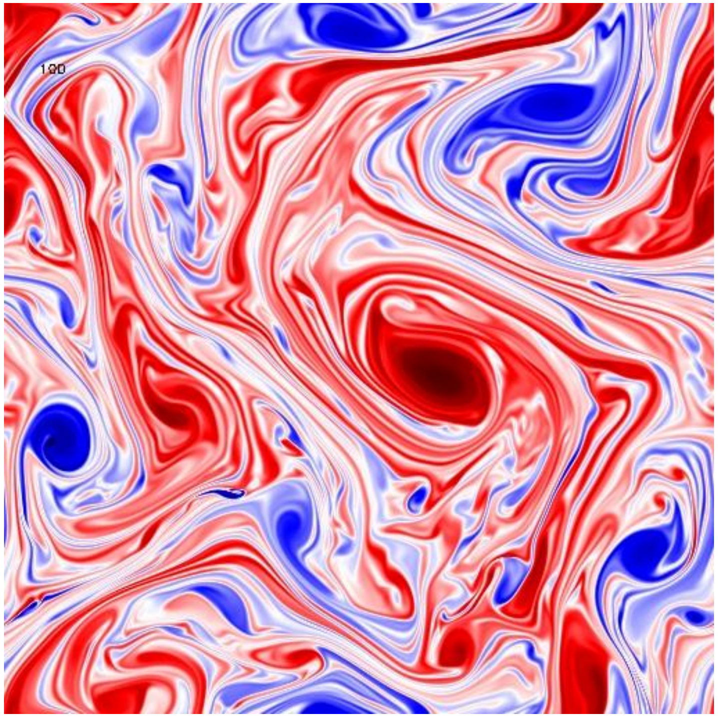
There are many similarities:

Stability theorems which rely on symmetry properties of inversion
E/Z transfer arguments – formation of coherent vortices
Turbulence scaling arguments

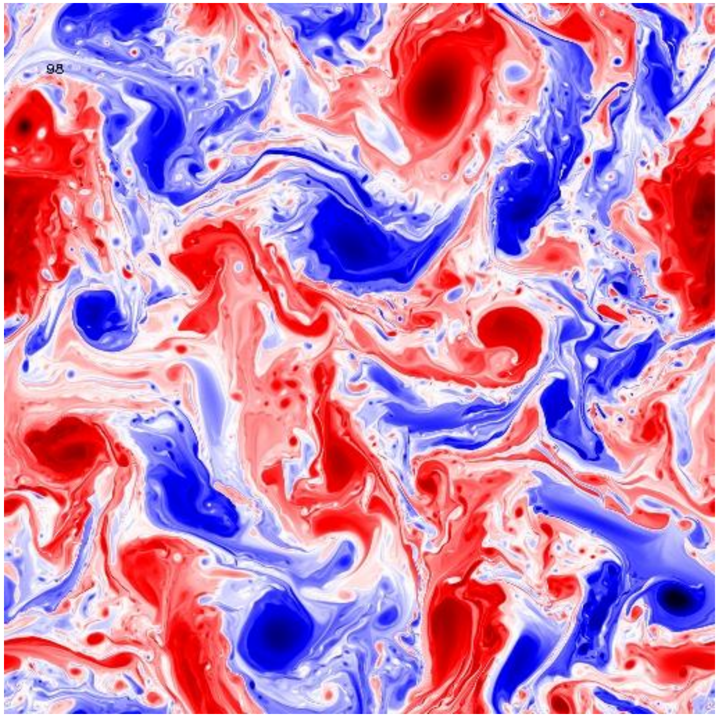
Comparison to Euler equations - II

Snapshots of freely decaying turbulence:

2-d Euler



SQG



Both have: large coherent vortices plus complicated small scale structure

But: SQG vortices are less tightly bound and small scale structure is more ‘messy’

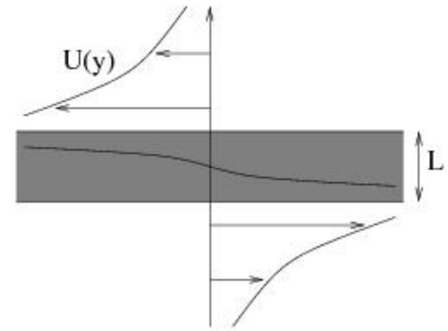
SQG filament instability problem

Calculated by Juckes (1995).

Basic state:

$$\theta(y) = \begin{cases} \theta_0 & |y| < L/2 \\ 0 & |y| > L/2 \end{cases}$$

$$U(y) = \frac{\theta_0}{\pi} \log\left|\frac{y-L/2}{y+L/2}\right|$$



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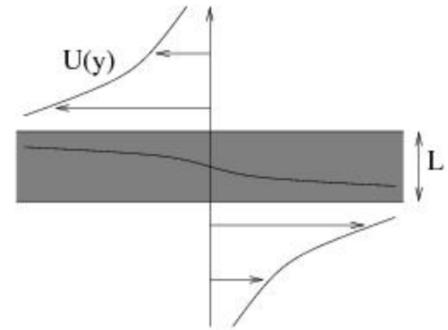

Perturbation:

$$(\eta_1, \eta_2) = \hat{\boldsymbol{\eta}}(t) e^{ikx}$$

$$i \frac{d\hat{\boldsymbol{\eta}}}{dt} = \frac{\theta_0}{L} \begin{pmatrix} P(\kappa) & I(\kappa) \\ -I(\kappa) & -P(\kappa) \end{pmatrix} \hat{\boldsymbol{\eta}}$$

where $\kappa = kL$

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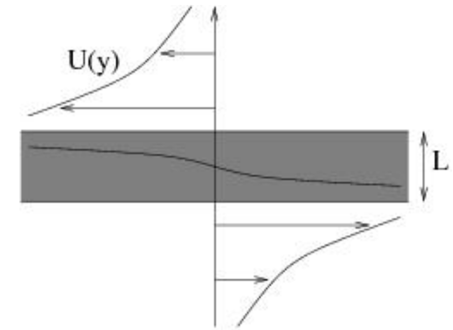

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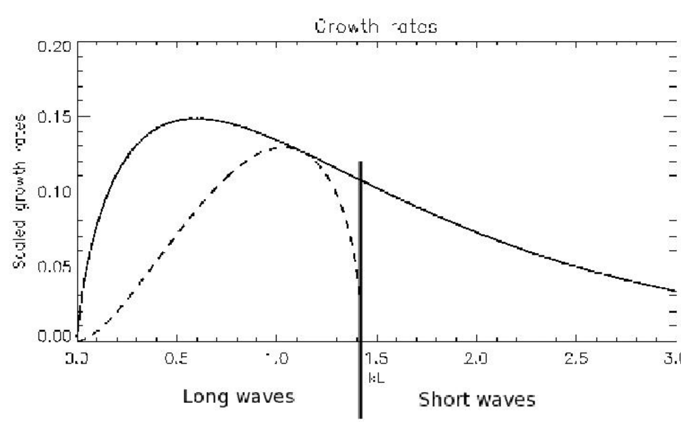
Growth rates:

(use RMS wave-slope norm:

$$\mathcal{A}_{max} = \kappa |\hat{\boldsymbol{\eta}}|$$

)



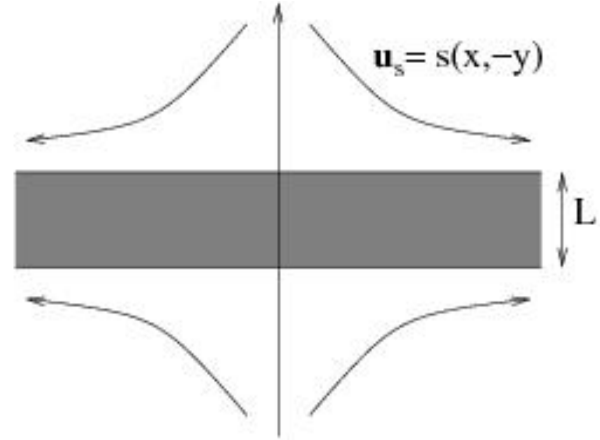
Growth rates proportional to θ_0/L

Normal mode growth rate is

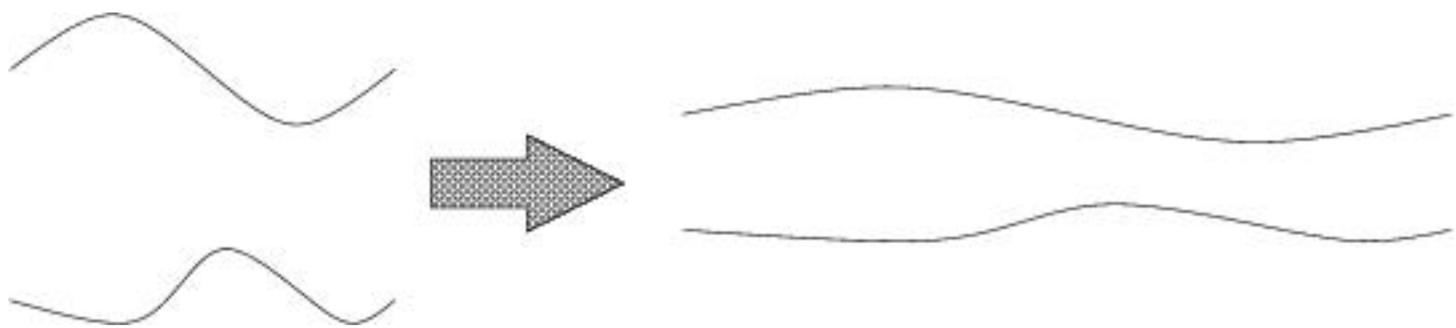
$$\sigma_{nm} = \det F = \frac{\theta_0}{L} \sqrt{I^2 - P^2}$$

Kinematic effects of straining

Consider a filament under a pure straining flow:



The filament will be stretched and squashed exponentially:



$$L = L_0 e^{-st}, \; k = k_0 e^{-st}, \; \kappa = \kappa_0 e^{-2st}$$

For vorticity filaments in the Euler equations this process can stabilise filaments (Dritschel et al 1991).

What happens in the SQG case?

Dimensional argument

2-d Euler equations

Vorticity ξ has dimension of $(\text{time})^{-1}$

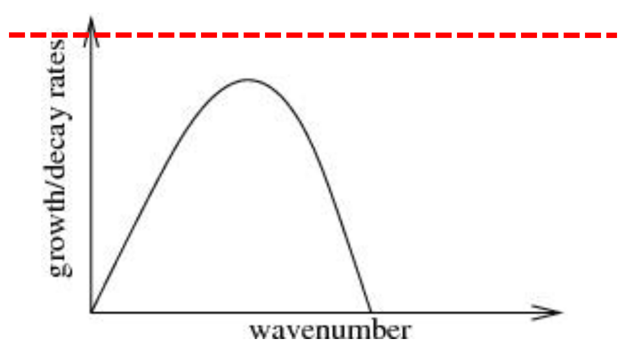
So the only non-dimensional parameter is

$$S = \frac{s}{\xi_0}$$

Dritschel et al show that vorticity filaments are stabilised when

$$s > C\xi_0 \text{ i.e. } S > C$$

Pictorially:



SQG equations

Temperature θ has dimension of (length)$(\text{time})^{-1}$

So the non-dimensional parameter here is

$$S = \frac{sL}{\theta_0}$$

Note that, since *L* decreases in time, *S* decreases exponentially to zero.

This suggests that SQG filaments cannot be stabilised by straining over a long time period.

We confirm this result via a linear stability analysis.

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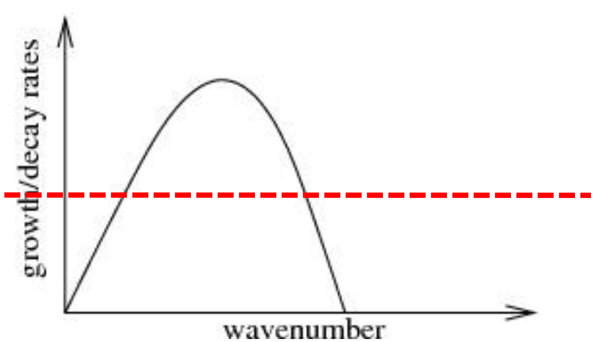

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Initial value problem - I: Some example integrations

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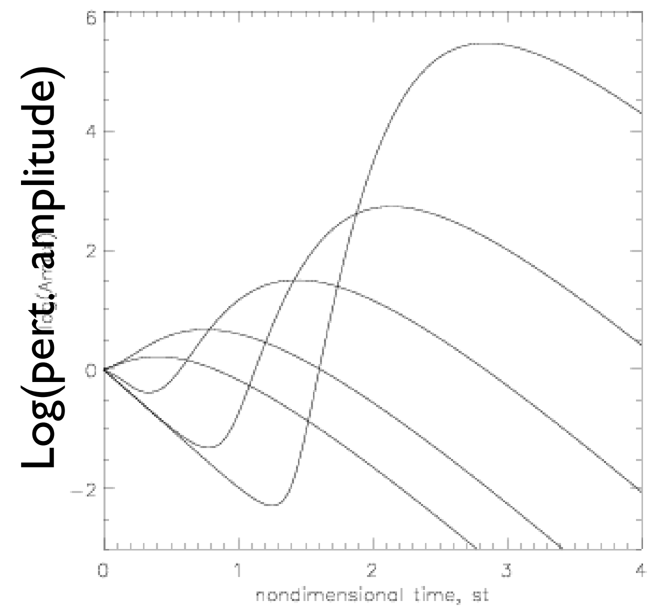
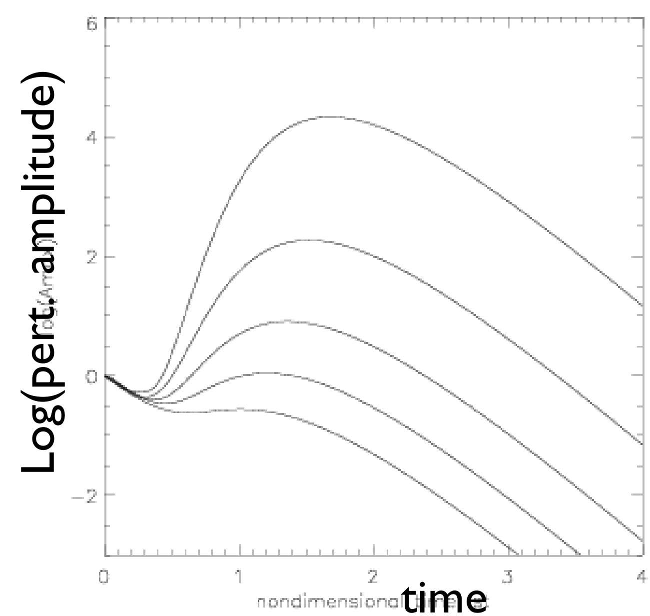
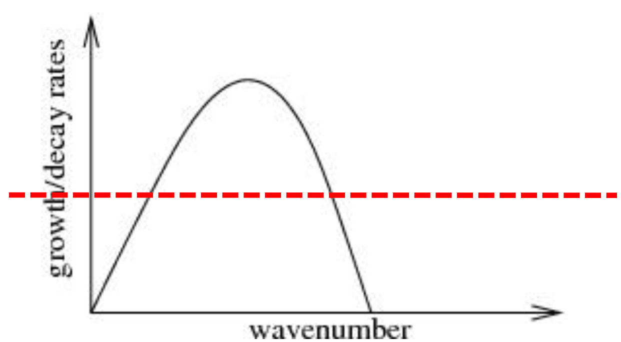


Increasing k_0

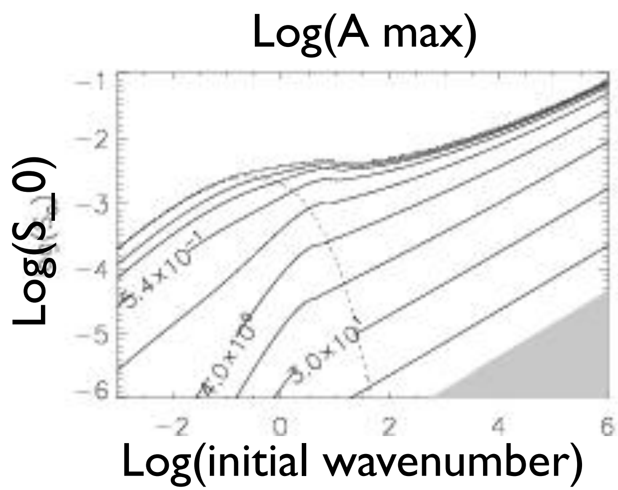


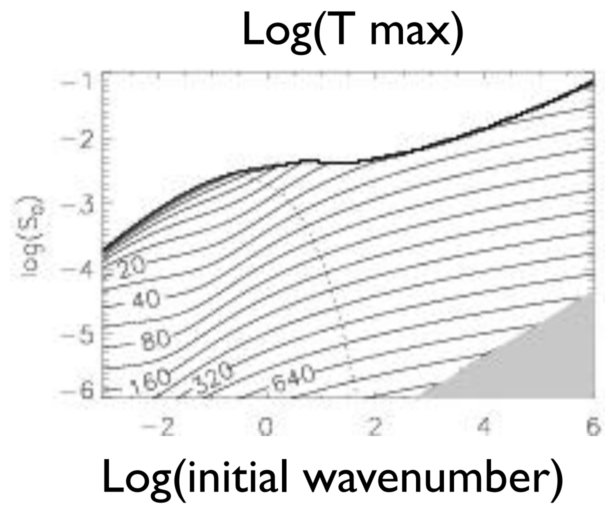
Increasing strain

Pictorially:



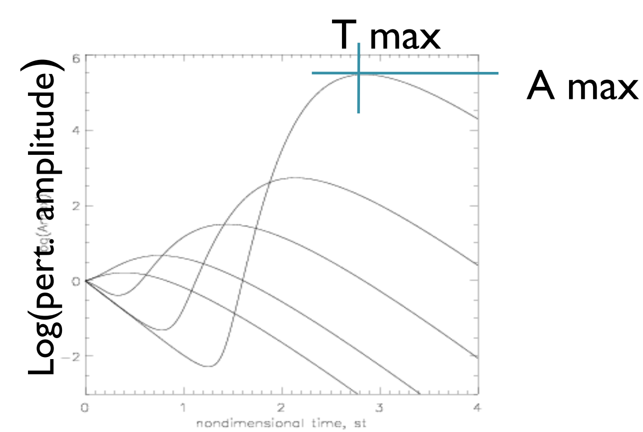
Initial value problem - II: Maximum value approach



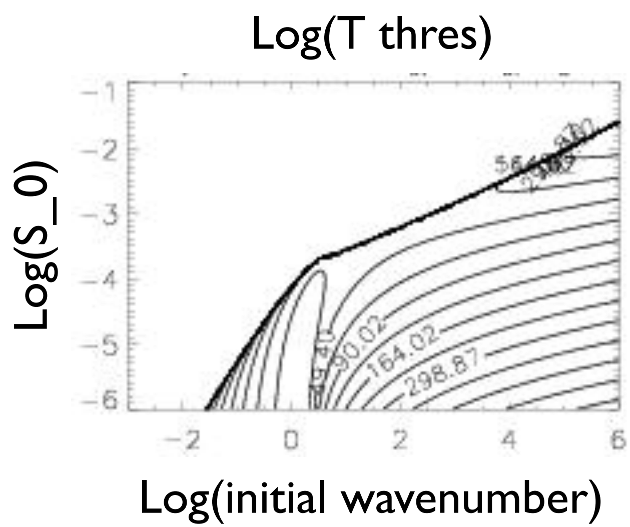


•For all strain rates there are unstable perturbations

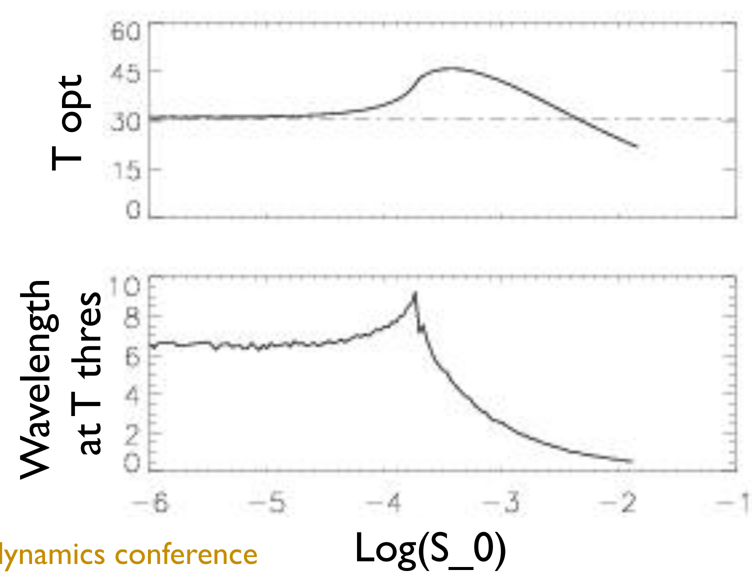
•For all perturbations there are stabilising strain values

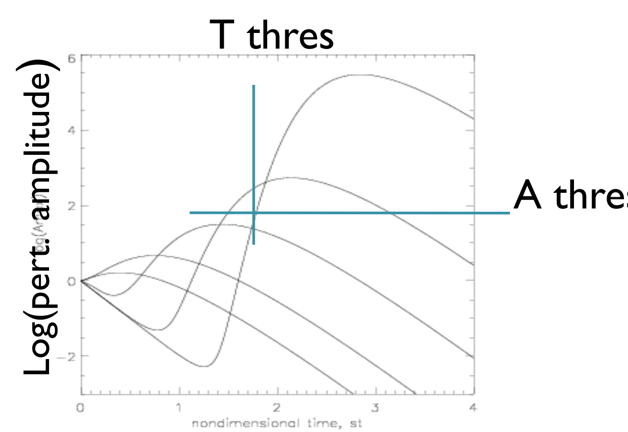


Initial value problem – III: Threshold value approach



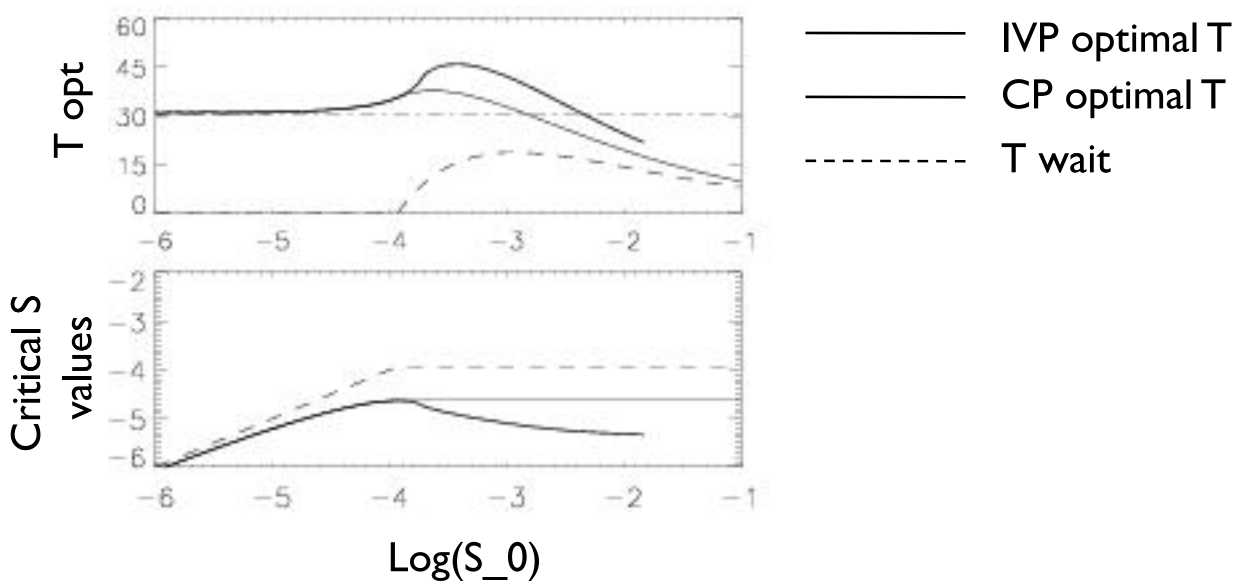
For all strain rates there is an optimal wavenumber for achieving threshold growth





Initial value problem – IV: Continued perturbations approach

It is possible that a perturbation started after the start of an IVP integration will reach threshold amplitude earlier than the initial perturbation



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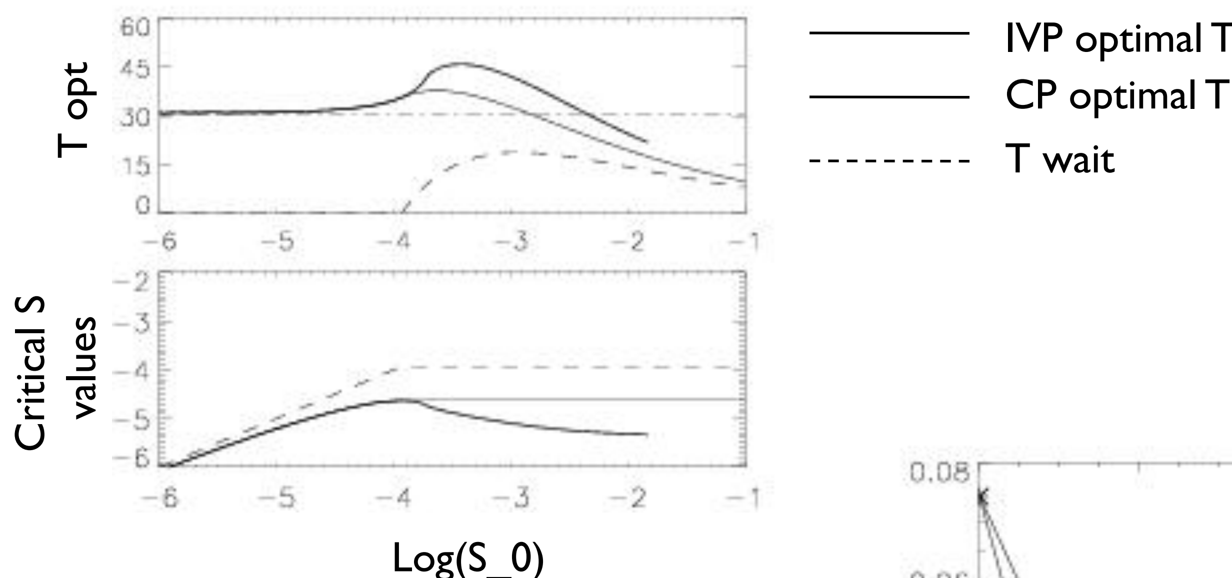
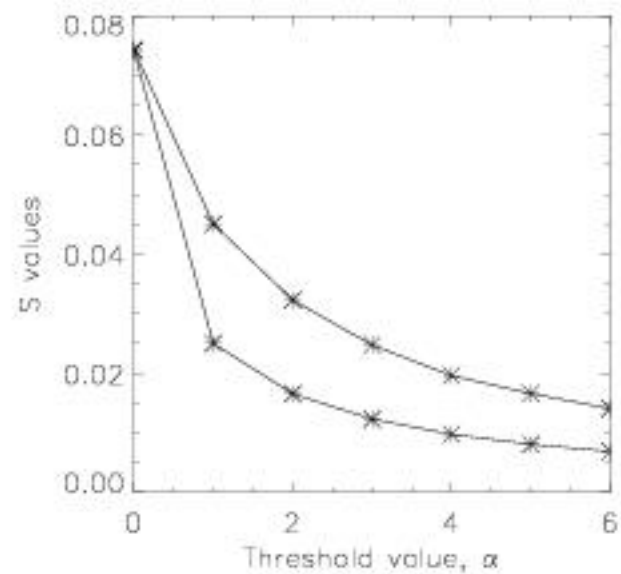


Other threshold values:



Conclusions

- Temperature filaments under SQG dynamics are strongly affected by an external straining flow.

- Whilst the parameter $S = \frac{sL}{\theta_0}$ > 0.0743 ... a filament is stabilised by the strain.

- However, the strain also causes S to decrease exponentially and eventually instability will occur.

- For 'significant' growth to occur a value of S > 0.02 may be more appropriate.

- In either case, at the time when $S=S_c$ the width of the filament is

$$L = \frac{\theta_0}{s} S_c$$

which is independent of the initial condition of the filament (!)

- Therefore the size of small scale vortices in SQG turbulence is strongly linked to the strength of the straining flow in which they were formed.

Thank you for your attention!

References

Dritschel et al (1991): The stability of a two-dimensional vorticity filament under uniform strain. *J. Fluid. Mech.* **230**, pp. 647-665.
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Schär and Davies (1990): An Instability of Mature Cold Fronts. *J. Atmos. Sci.* **47**, pp. 929-950.

SQG filament under uniform shear