Surface quasi-geostrophic vortices

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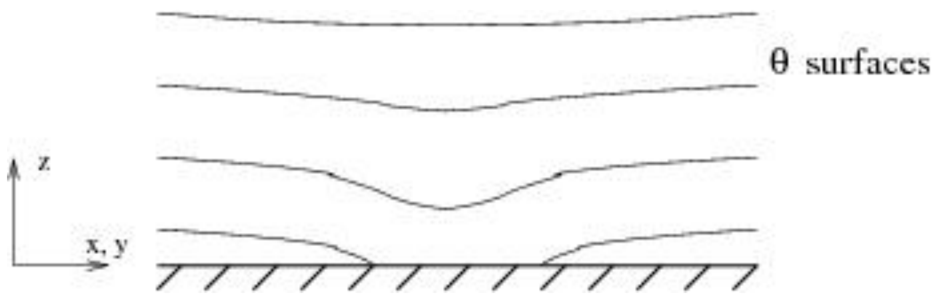
What is surface quasi-geostrophic (SQG) dynamics?

Models a rotating stratified fluid near a horizontal boundary

Consider quasi-geostrophic motion with uniform interior potential vorticity:

E.g.



QGPV anomaly is zero: $q = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{f^2}{N^2} \frac{\partial^2}{\partial z^2} \right) \psi = 0$ in z>0,

with potential temperature $\theta = \frac{\partial \psi}{\partial z}$ conserved at the boundary: $\frac{D\theta}{Dt} = 0$ at z=0.

What is surface quasi-geostrophic (SQG) dynamics?

This is a two-dimensional advection equation with inversion given by

$$\psi(\mathbf{x}) = -\frac{1}{2\pi}\iint \frac{\theta(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\,\mathrm{d}^2\mathbf{x}' \quad \text{or} \quad \hat{\psi}(\mathbf{k}) = -\frac{\hat{\theta}(\mathbf{k})}{|\mathbf{k}|}$$

Compare with the 2-d Euler equations:

Vorticity $q = \nabla^2\psi$ is conserved

with inversion is given by

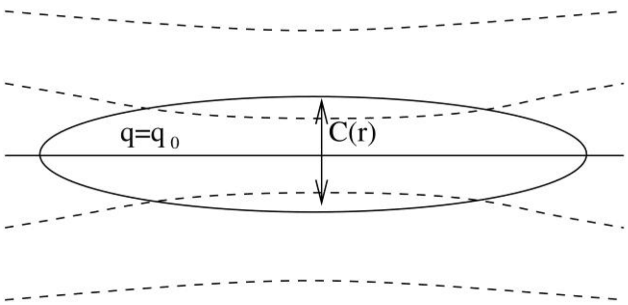
$$\psi(\mathbf{x}) = \frac{1}{2\pi}\iint q(\mathbf{x}')\log|\mathbf{x}-\mathbf{x}'|\,\mathrm{d}^2\mathbf{x}' \quad \text{or} \quad \hat{\psi}(\mathbf{k}) = -\frac{\hat{q}(\mathbf{k})}{|\mathbf{k}|^2}$$

Bretherton (1966) interpretation

Formally, the SQG system is equivalent to the dynamics of a QGPV δ-function in the vertical:

$$q = 2\theta(x,y)\delta(z)$$

This falls out of the maths. To see physically, consider the following distribution of QGPV:

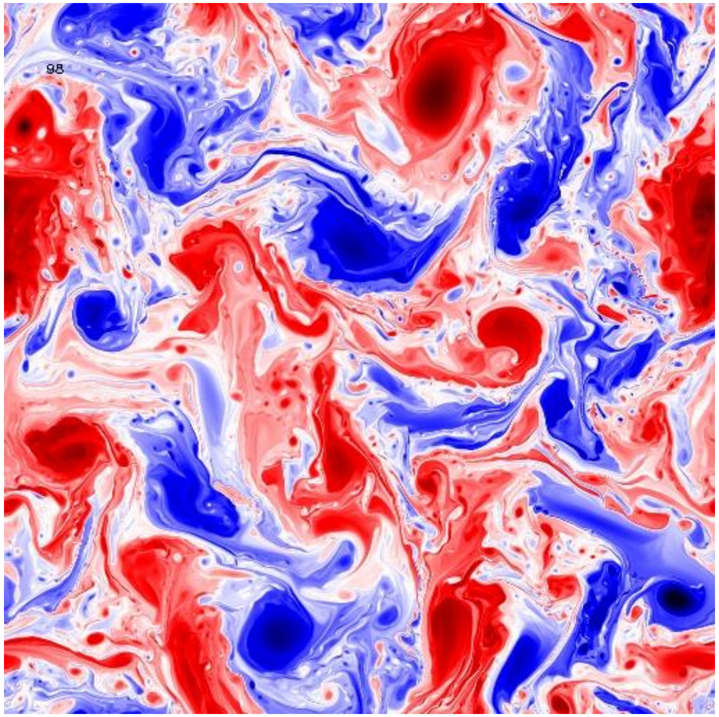


$$q = \xi + \theta_z$$

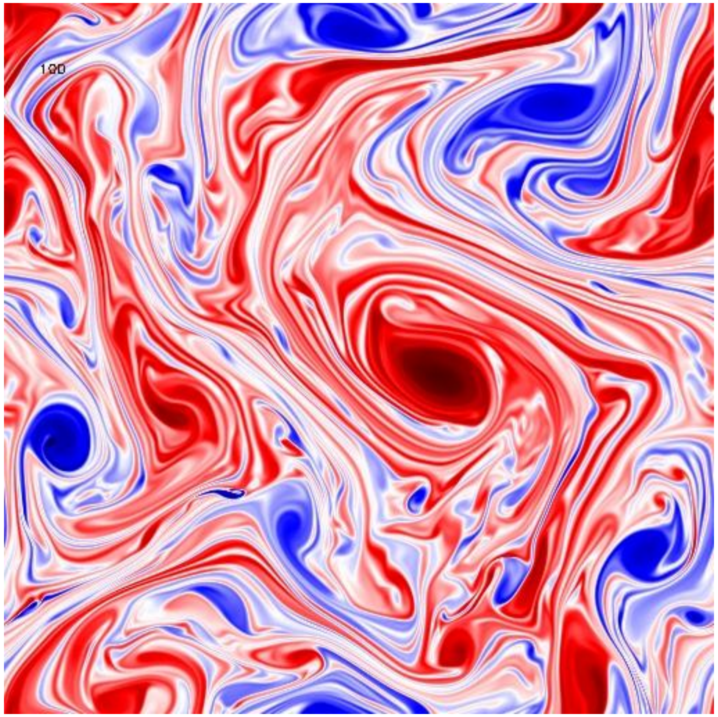
Note also for reference that the 2-d Euler system is equivalent to the dynamics of an infinitely deep QGPV distribution

Snapshots of freely decaying turbulence

SQG



2-d Euler



There are many similarities:

- Stability theorems which rely on symmetries of the inversion operator
- Energy/enstrophy transfer arguments
- Formation of coherent vortices
- Turbulence spectra, ...

Outline

I. Stable vortices

II. Unstable vortices

I

Stable vortices

-Ellipsoid solution

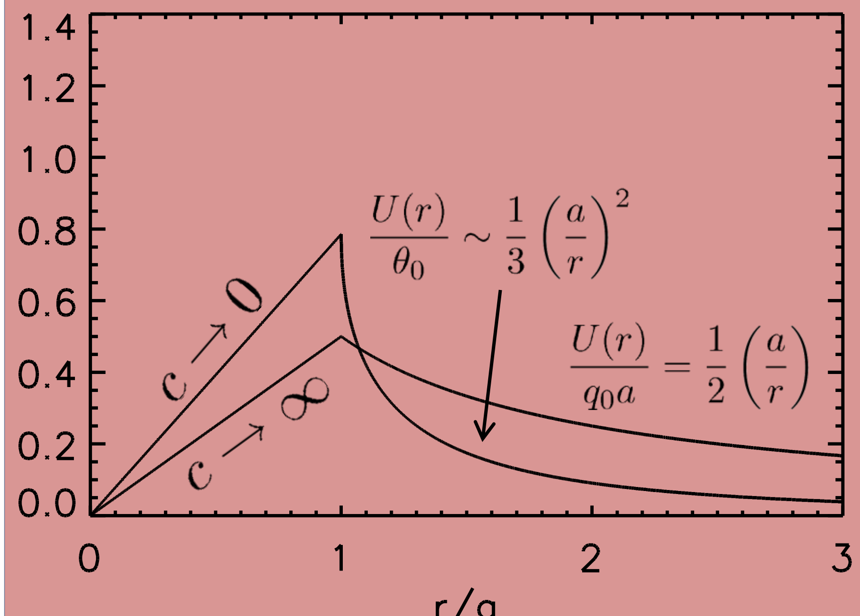
-Patch solution

Ellipsoid solution

In unbounded QGPV dynamics an ellipsoid of uniform PV is a steadily rotating solution:

$$q = \begin{cases} q_0 & \text{for} \quad \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 < 1 \\ 0 & \text{otherwise} \end{cases}$$

Velocity profiles (for a=b)



Two interesting limits:

$c \to \infty$ (q_0 fixed)
2-d Euler Rankine vortex

$c \to 0$ ($q_0 c$ fixed)
SQG non-uniform ellipse:

$$\theta = \begin{cases} \theta_0 \sqrt{1 - \left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2} & \text{for} \quad \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 < 1 \\ 0 & \text{otherwise} \end{cases}$$

Dritschel (2010) *Geophys. & Astro. Fluid Dyn.*

Patch solution – basic state

Another interesting vortex solution is given by a patch of uniform temperature:

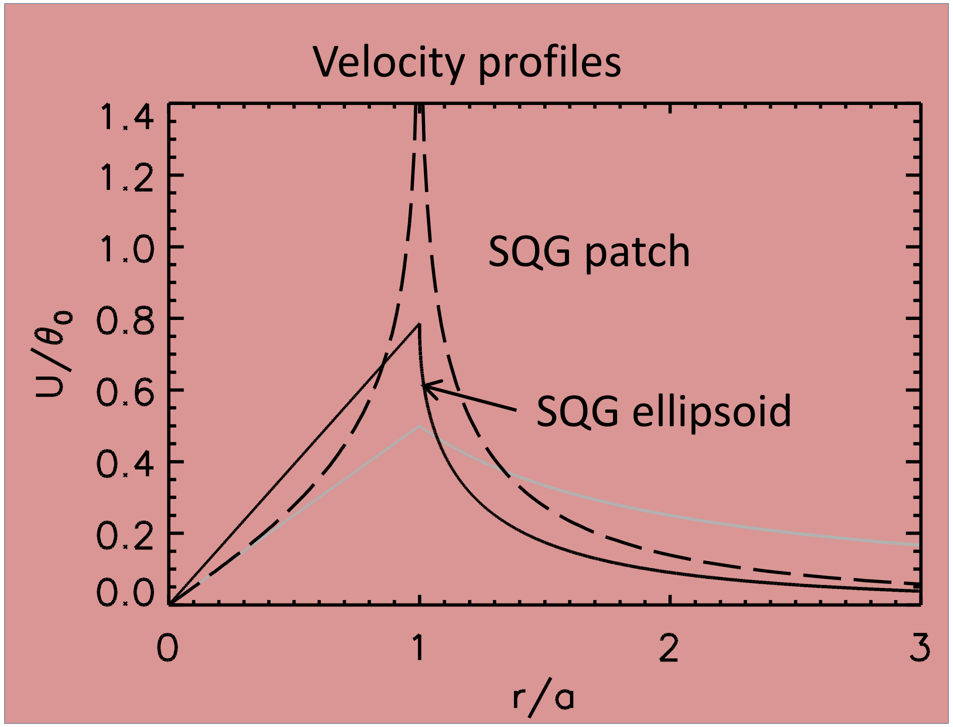
$$\theta(r) = \begin{cases} \theta_0 & \text{for} \quad r < a \\ 0 & \text{otherwise} \end{cases}$$

Azimuthal velocity field in terms of Bessel functions:

$$U(r) = \theta_0 \int_0^\infty \mathrm{J}_1(\kappa)\mathrm{J}_1(\kappa r/a)\,\mathrm{d}\kappa$$

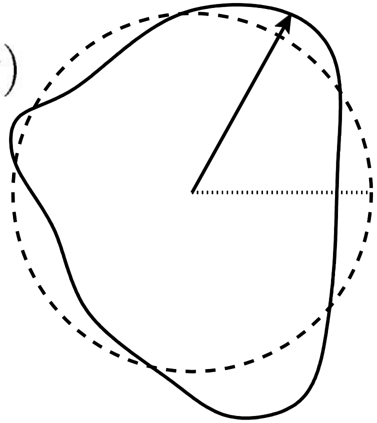
Compare to 2-d Euler case:

$$U(r) = q_0 a \int_0^\infty \mathrm{J}_1(\kappa)\mathrm{J}_1(\kappa r/a)\,\frac{\mathrm{d}\kappa}{\kappa}$$



Patch solution - perturbations

$$r = a + \eta(\psi, t)$$



The boundary of the SQG patch vortex supports perturbations:

We can derive a dispersion relation for linear perturbations using the contour dynamics formula:

$$\mathbf{u}(\mathbf{x}) = \theta_0 \oint \mathrm{G}(|\mathbf{x} - \mathbf{x}'|) \begin{pmatrix} \mathrm{d}x' \\ \mathrm{d}y' \end{pmatrix}$$

And continuity at the patch boundary:

$$\frac{\partial \eta}{\partial t} = u_{\mathrm{r}} - \frac{u_\psi}{a + \eta} \frac{\partial \eta}{\partial \psi}$$

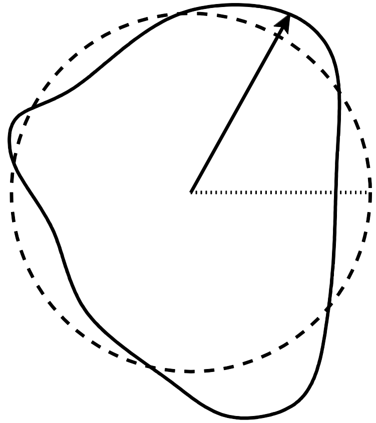
The result is:

$$\omega_n = \frac{\theta_0 n}{a\pi} \sum_{i=2}^{n} \frac{1}{i - 1/2}$$

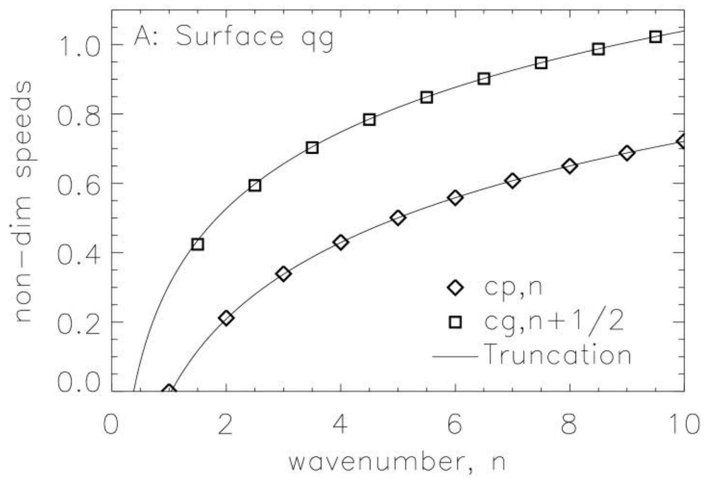
Harvey & Ambaum (2010) *Geophys. & Astro. Fluid Dyn.*

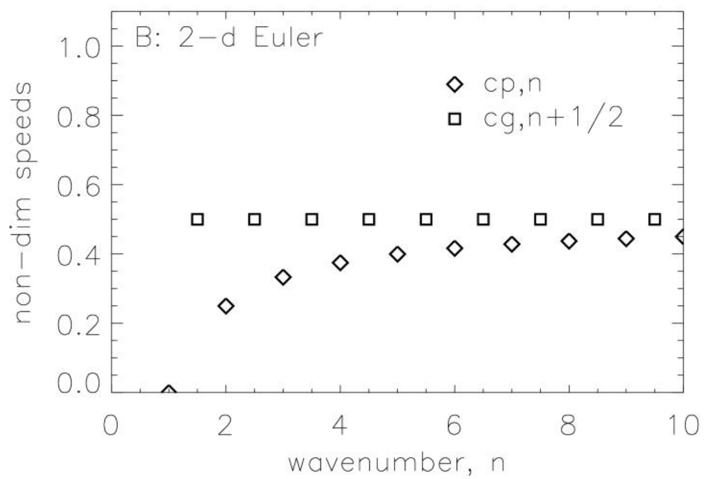
Patch solution – comparison to 2-d Euler

Surface QG patch result: $$\omega_n = \frac{\theta_0 n}{a\pi} \sum_{i=2}^{n} \frac{1}{i - 1/2}$$



The corresponding 2-d Euler result is: $$\omega_n = \frac{q_0 n}{2} \left(1 - \frac{1}{n}\right)$$



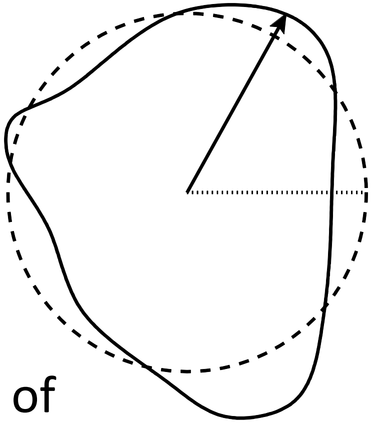


Harvey & Ambaum (2010) *Geophys. & Astro. Fluid Dyn.*

Patch solution – consequences

Surface QG patch result:

$$\omega_n = \frac{\theta_0 n}{a\pi} \sum_{i=2}^{n} \frac{1}{i - 1/2}$$



Influence of a background flow:

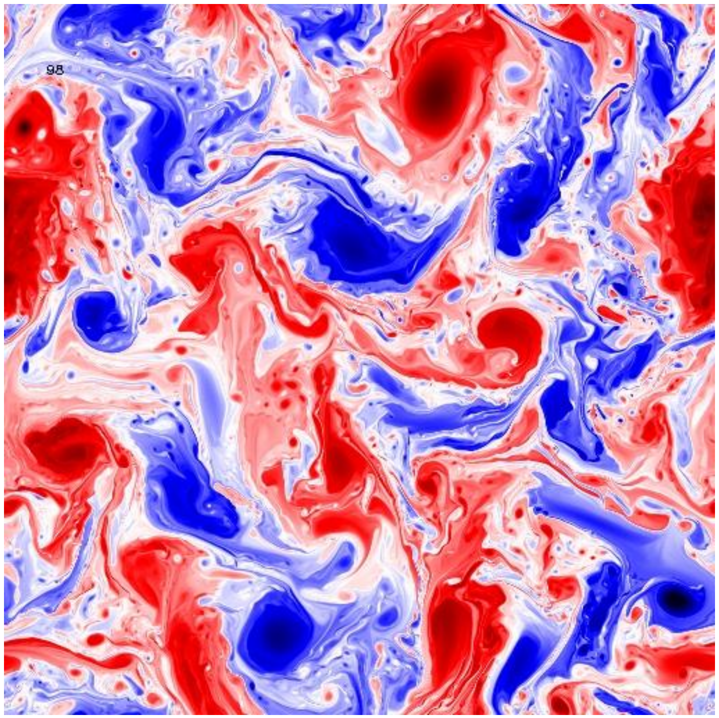
For a pure straining flow, $(u, v) = s(x, -y)$ in the limit of small straining, $s \ll \theta_0/a$ the deformation will be small so will satisfy the dispersion relation.

The n=2 mode can propagate against the induced rotation of the strain to give a steady state if the perturbation amplitude satisfies

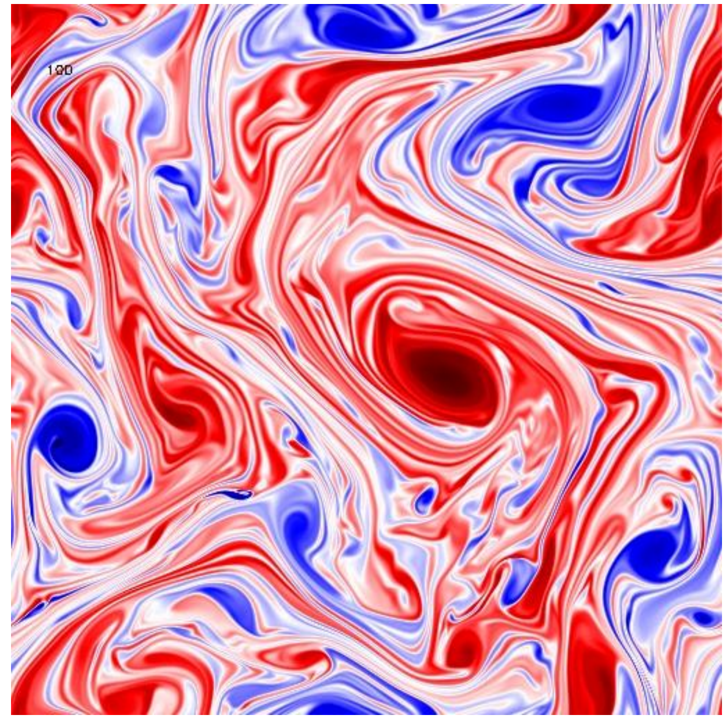
$$\frac{\eta_0}{a} = \frac{3\pi}{4} \frac{as}{\theta_0}$$

Harvey & Ambaum (2010) *Geophys. & Astro. Fluid Dyn.*

SQG



2-d Euler



II

Unstable vortices

-Stability conditions

-Some smooth profiles

-2-step patch profiles

Stability conditions

Many 2-d Euler results on flow stability carry over to the SQG system

The most basic is the Rayleigh theorem

The polar coordinate form is

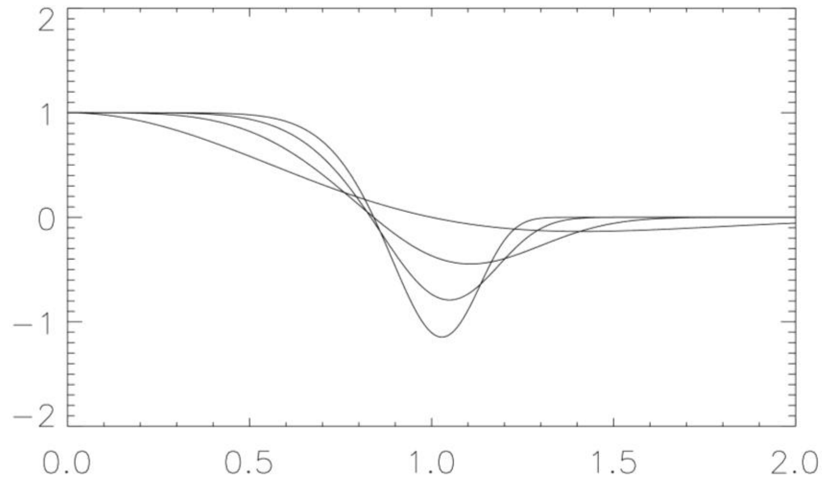
Linear perturbations , $\theta'(r, \psi, t)$, to a radially symmetric temperature profile $\Theta(r)$ satisfy the constraint

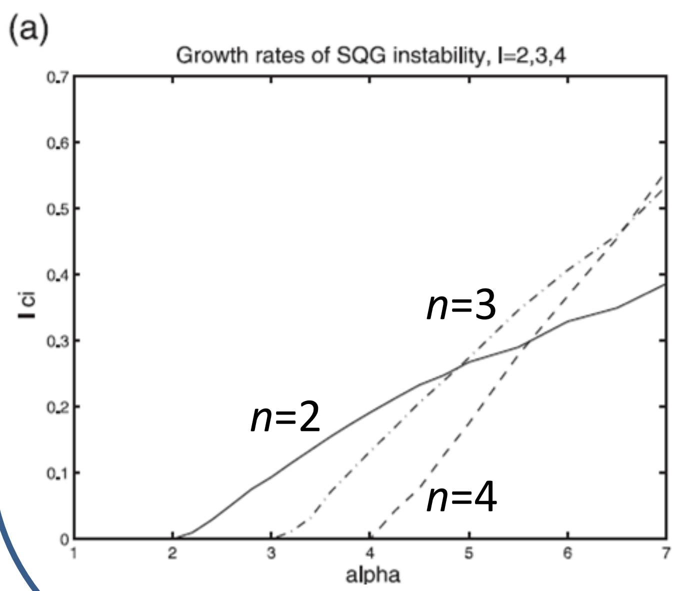
$$\frac{\mathrm{d}}{\mathrm{d}t} \iint \theta'^2 \frac{r}{\mathrm{d}\Theta/\mathrm{d}r} r \, \mathrm{d}r \, \mathrm{d}\psi = 0$$

Analogue to 2-d Euler Rayleigh theorem.
This holds for a general self-adjoint inversion operator

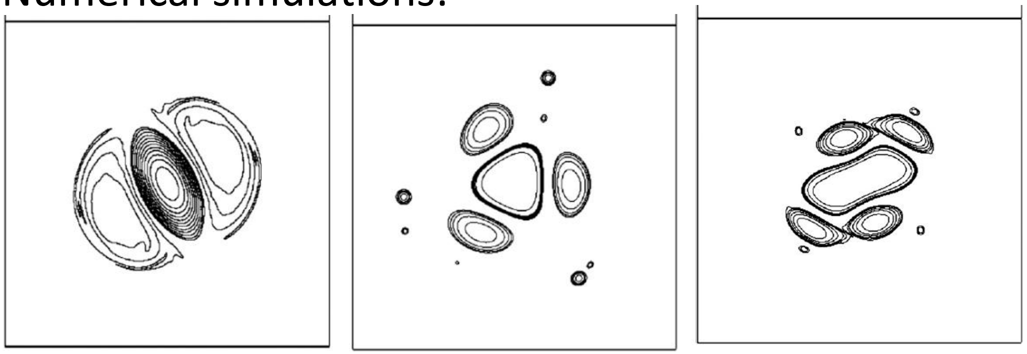
Some smooth profiles

$$\Theta(r) = \theta_0 \left(1 - \frac{\alpha}{2} r^{\alpha}\right) e^{-r^{\alpha}}$$





Numerical simulations:



a=3 *a*=6 *a*=7

Carton (2009)
J. Atmos. Sci.

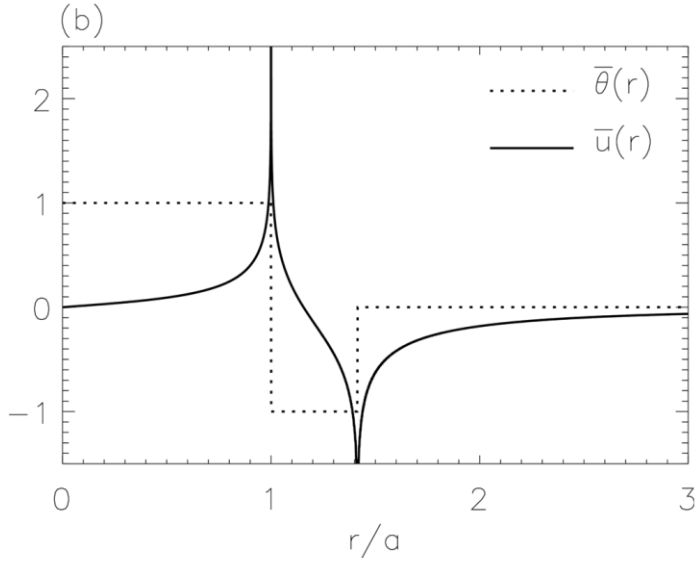
2-step patch profiles – basic state

Consider a 2-step patch profile:

$$\theta = \begin{cases} \theta_0 & \text{for } r < a \\ \theta_1 & \text{for } a < r < b \\ 0 & \text{for } r > b \end{cases}$$

The induced velocity field is a linear combination of single-patch inversions:

$$U(r) = (\theta_0 - \theta_1) \int_0^\infty \mathrm{J}_1(\kappa)\mathrm{J}_1(\kappa r/a)\,\mathrm{d}\kappa + \theta_1 \int_0^\infty \mathrm{J}_1(\kappa)\mathrm{J}_1(\kappa r/b)\,\mathrm{d}\kappa$$



2-step patch profiles – perturbations

We can derive the linear dynamics as for the single patch problem.

$$\theta = \begin{cases} \theta_0 & \text{for } r < a \\ \theta_1 & \text{for } a < r < b \\ 0 & \text{for } r > b \end{cases}$$

Perturb boundaries to $r = a + \eta(\psi, t)$ and $r = b + \nu(\psi, t)$

Linearise the contour dynamics formulae

The evolution of each Fourier mode is then given by

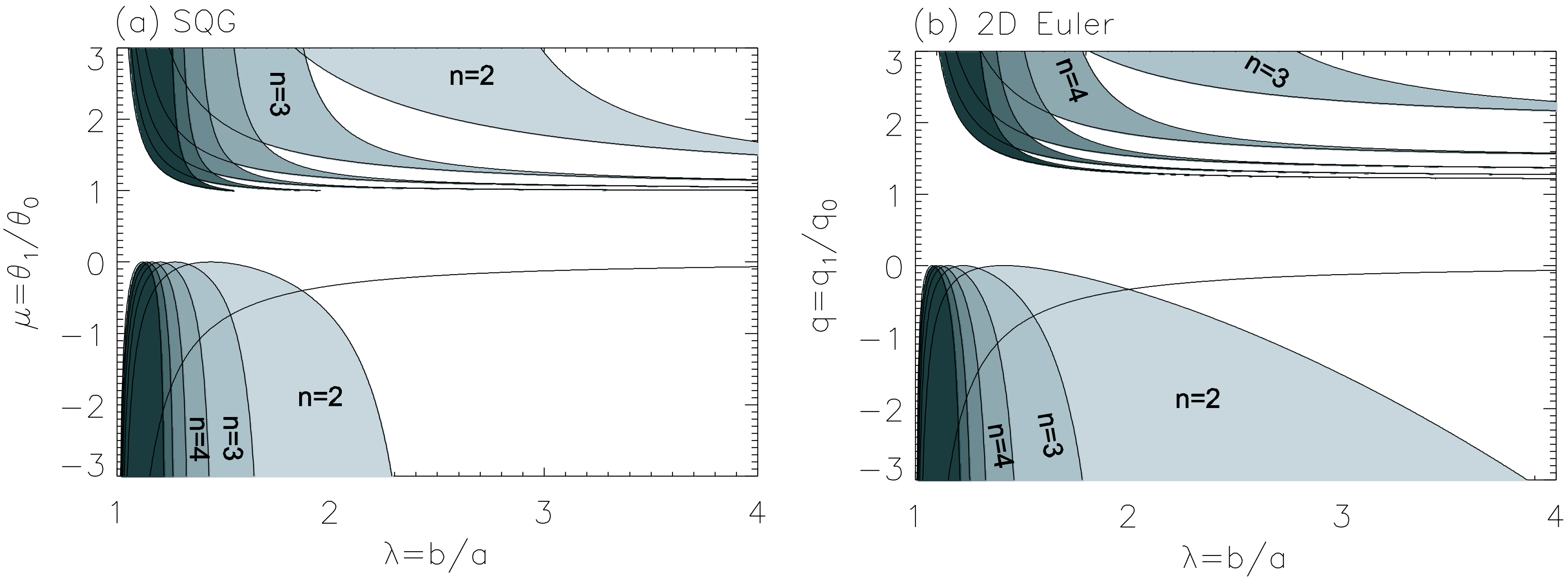
$$i\frac{d}{dt}\begin{pmatrix} \hat{\eta} \\ \hat{\nu} \end{pmatrix} = \frac{\theta_0 n}{a}\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} \hat{\eta} \\ \hat{\nu} \end{pmatrix}$$

with $A, B, C \text{ and } D = func(b/a, \theta_1/\theta_0)$

2-step patch profiles – normal modes

$$\theta = \begin{cases} \theta_0 & \text{for } r < a \\ \theta_1 & \text{for } a < r < b \\ 0 & \text{for } r > b \end{cases}$$

Normal mode boundaries of stability:



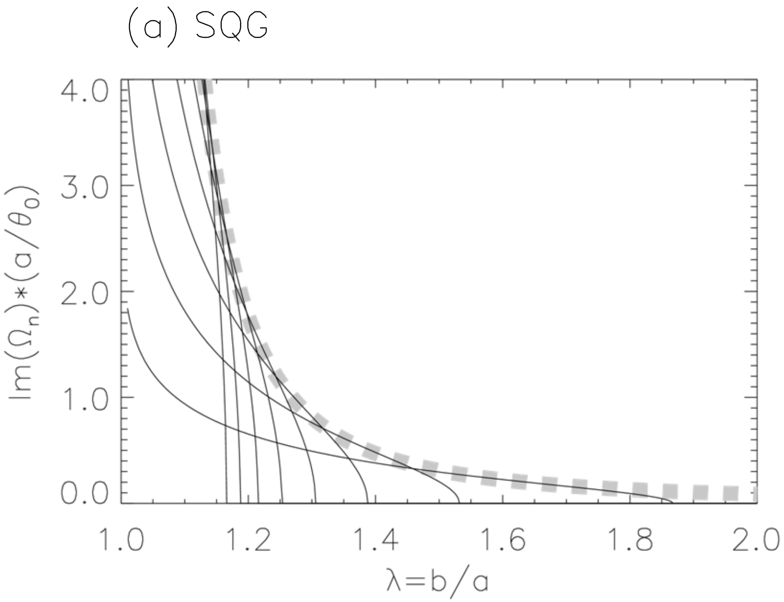
Flierl (1988)
J. Fluid Mech.

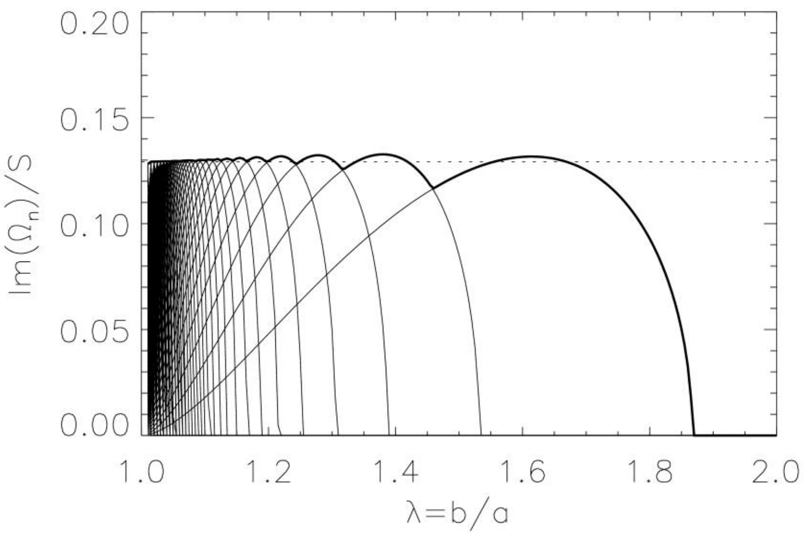
2-step patch profiles – normal modes

Isolated vortices: $$\int_0^\infty \theta(r) r \, \mathrm{d}r = 0$$

$$\theta = \begin{cases} \theta_0 & \text{for } r < a \\ \theta_1 & \text{for } a < r < b \\ 0 & \text{for } r > b \end{cases}$$

Normal mode growth rates:



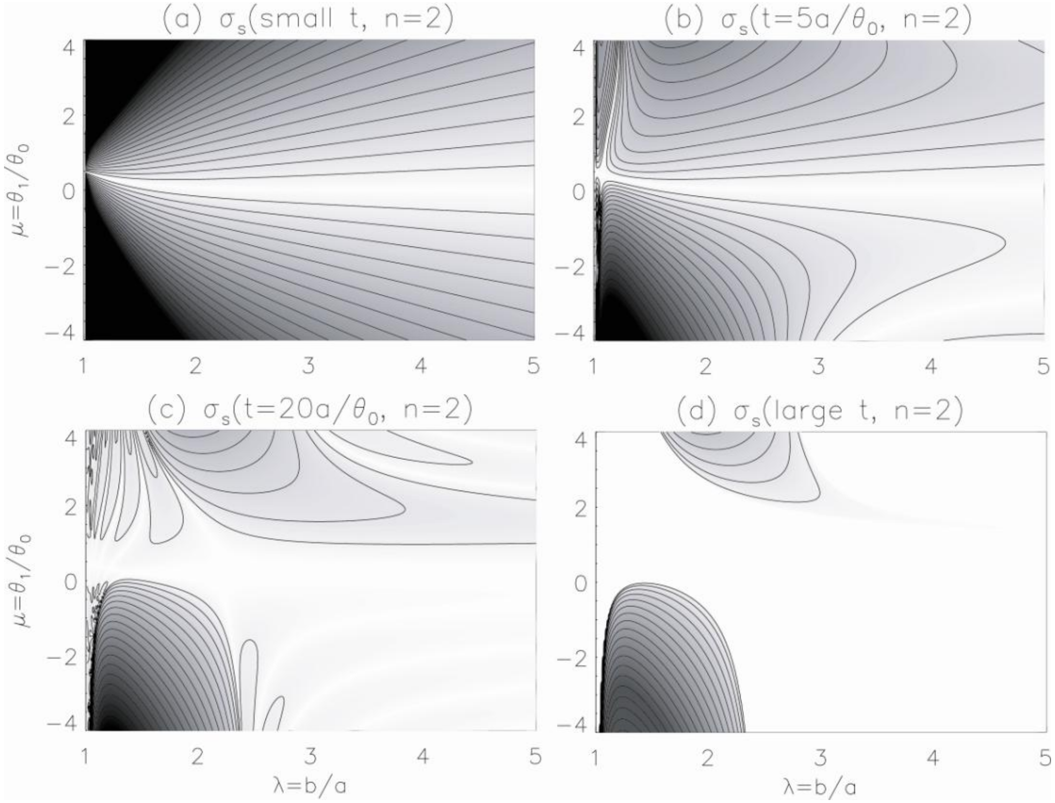


An alternative filament-like scaling is $$S = \frac{\sqrt{|(\theta_0 - \theta_1)\theta_1|}}{b - a}$$

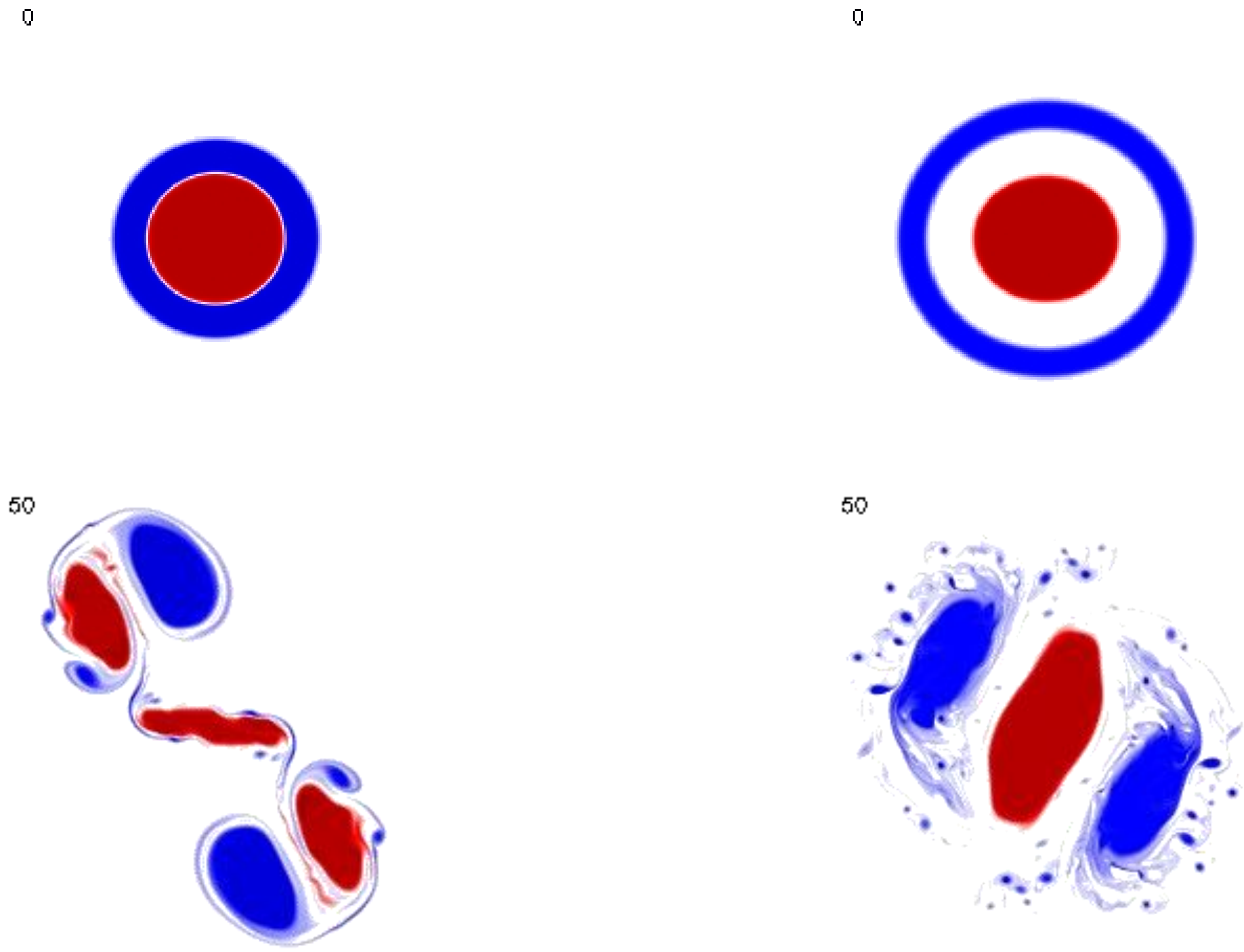
2-step patch profiles – non-modal solution

Most solutions are non-modal

Analyse with the *equivalent growth rate*, $\sigma = \frac{\log \mathcal{A}(t)}{t}$



2-step patch profiles – nonlinear simulations



2-step patch profiles – comparison to ellipsoid solution

Summary