Nonlinear error dynamics for cycled data assimilation methods

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Outline

Outline

Dynamical system

- Dynamical system: nonlinear dynamical flow and a linear measurement equation.
- Let $\mathcal{M}_k^{(t)} : \mathbf{X} \rightarrow \mathbf{X}$ and $\mathcal{M}_k : \mathbf{X} \rightarrow \mathbf{X}$ be a true nonlinear model operator and a modelled nonlinear model operator respectively.
- All operators are mapping a state $\mathbf{x}_{k-1}^{(t)} \in \mathbf{X}$ discretely onto its state $\mathbf{x}_k^{(t)}$ for $k \in \mathbb{N}_0$ for the Hilbert space $(\mathbf{X}, \|\cdot\|_{\mathbf{X}})$.
- The operator $\mathcal{M}_k : \mathbf{X} \rightarrow \mathbf{X}$ is modelled, such that

$$\mathcal{M}_k\left(\mathbf{x}_k^{(t)}\right) = \mathcal{M}_k^{(t)}\left(\mathbf{x}_k^{(t)}\right) + \zeta_{k+1}, \qquad (1)$$

where ζ_{k+1} is some additive noise which we call *model error* and is bounded by some constant $\upsilon > 0$ for all time t_k for $k \in \mathbb{N}_0$.

Observations

- Let H be an injective linear time-invariant compact observation operator, such that $H : \mathbf{X} \to \mathbf{Y}$, for Hilbert spaces $(\mathbf{X}, \|\cdot\|_{\mathbf{X}})$ and $(\mathbf{Y}, \|\cdot\|_{\mathbf{Y}})$.
- Let $\mathbf{y}_k^{(t)} \in \mathbf{Y}$ be the true observations (measurements) located at discrete times t_k linearly, such that

$$\mathbf{y}_k^{(t)} = H^{(t)}\mathbf{x}_k^{(t)} = H\mathbf{x}_k^{(t)} = \mathbf{y}_k - \boldsymbol{\eta}_k, \tag{2}$$

  where $\boldsymbol{\eta}_k$ is some additive noise that we call the *observation error* and is bounded by some constant $\delta > 0$ for all time t_k for $k \in \mathbb{N}_0$.
- It is possible to carry through the same analysis using a noise term on H modelled,

$$\left(H - H^{(t)}\right)\mathbf{x}_k^{(t)} = \boldsymbol{\omega}_k. \tag{3}$$

A couple of definitions

Definition (p.10 in Kress 1999)

An inner product space, which is complete with respect to the norm

$$\|\mathbf{x}\| := \langle \mathbf{x}, \mathbf{x} \rangle^{1/2}, \quad (4)$$

for all $\mathbf{x} \in \mathbf{X}$, is called a Hilbert space.

Definition (Definition 7.1 in Rynne and Youngson 2007)

Let $\mathbf{X}$ and $\mathbf{Y}$ be normed space. An operator $H \in L(\mathbf{X}, \mathbf{Y})$ is compact if, for any bounded sequence $(\mathbf{x}_n)$ in $\mathbf{X}$, the sequence $(H\mathbf{x}_n)$ in $\mathbf{Y}$ contains a convergent subsequence.

Aim

- Develop theory demonstrating the asymptotic stability of a cycled data assimilation scheme with an ill-posed observation operator in a nonlinear infinite dimensional setting.
- Work within the framework of data assimilation scheme that employ static covariances, such as three dimensional variational data assimilation (3DVar).
- Here we extend previous linear results from *R W E Potthast, A J F Moodey, A S Lawless and P J van Leeuwen 2012*.

Stability

- We define the *analysis error* as the difference between the analysis and the true state of the system, such that

$$\mathbf{e}_k := \mathbf{x}_k^{(a)} - \mathbf{x}_k^{(t)}, \tag{5}$$

  where $\mathbf{x}_k^{(t)}$ represents the true state of the system at time t_k for $k \in \mathbb{N}_0$.
- We will call a data assimilation scheme *stable* if given some constant $C > 0$

$$\|\mathbf{e}_k\|_{\mathbf{X}} \leq C \tag{6}$$

  as $k \to \infty$ with some appropriate norm $\|\cdot\|_{\mathbf{X}}$.
- The analysis error in other fields is known as state reconstruction error, observer error, estimation error, etc.

Outline

Operator equation

- Mathematically, we interpret the data assimilation task as seeking $\mathbf{x}_k^{(a)}$ at every assimilation step t_k, $k \in \mathbb{N}_0$ to solve an operator equation of the first kind

$$H\mathbf{x}_k^{(a)} = \mathbf{y}_k. \tag{7}$$

- This operator equation represents a Fredholm integral equation of the first kind and is ill-posed when the dimension of the state space $\mathbf{X}$ is infinite.

Ill-posedness

- Hadamard defined that a well-posed problem must satisfy:
  1. There exists a solution to the problem (existence).
  2. There is at most one solution to the problem (uniqueness).
  3. The solution depends continuously on the data (stability).

  If a problem does not satisfy all three of these condition, it is *ill-posed* in the sense of *Hadamard*.
- Nashed defined that an operator equation is called well-posed if
  1. the set of observations is a closed set, that is if $H(\mathbf{X})$ is closed.

  If a problem does not satisfy this property, then it is *ill-posed* in the sense of *Nashed* (*Nashed 1981, Nashed 1987*).

Ill-posed observation operator

Theorem (Theorem 15.4 in *Kress 1999*)

Let $\mathbf{X}$ and $\mathbf{Y}$ be normed spaces and $H \in L(\mathbf{X}, \mathbf{Y})$ be a compact operator. If $\mathbf{X}$ has infinite dimension then H cannot have a bounded inverse and the operator equation of the first kind is ill-posed.

- Regularization methods exist to provide a stable approximate solution to the ill-posed problem.
- *Tikhonov-Phillips regularization* shifts the the eigenvalues of the operator H^*H by a *regularization parameter* α, where H^* is the adjoint to the operator H.

Cycled Tikhonov-Phillips regularization

Definition

Given measurements $\mathbf{y}_k \in \mathbf{Y}$ for $k \in \mathbb{N}_0$ and an initial guess $\mathbf{x}_k^{(b)}$, the objective of cycled Tikhonov-Phillips regularization is to seek an estimate, $\mathbf{x}_k^{(a)}$ that minimises the functional,

$$\mathcal{J}^{(CTP)}(\mathbf{x}_k) := \alpha \left\| \mathbf{x}_k - \mathbf{x}_k^{(b)} \right\|_{\ell^2}^2 + \| \mathbf{y}_k - H\mathbf{x}_k \|_{\ell^2}^2 , \quad (8)$$

with respect to $\mathbf{x}_k$ for the ℓ^2 norm, where $\mathbf{x}_k^{(b)} = \mathcal{M}_{k-1}(\mathbf{x}_{k-1}^{(a)})$ and $\alpha > 0$.

Cycled Tikhonov-Phillips regularization

Theorem

If H is linear, the minimiser $\mathbf{x}_k^{(a)}$ to (8) is given by

$$\mathbf{x}_k^{(a)} = \mathcal{M}_{k-1}\mathbf{x}_{k-1}^{(a)} + \mathscr{R}_\alpha\left(\mathbf{y}_k - H\mathcal{M}_{k-1}\left(\mathbf{x}_{k-1}^{(a)}\right)\right), \quad (9)$$

given $\mathbf{x}_k^{(b)} = \mathcal{M}_{k-1}(\mathbf{x}_{k-1}^{(a)})$, where

$$\mathscr{R}_\alpha = (\alpha I + H^*H)^{-1}H^* \quad (10)$$

is known as the Tikhonov-Phillips inverse, with an adjoint H^* and a regularization parameter α. Here (9) is what we characterise as cycled Tikhonov-Phillips regularization.

Theorem

If H is linear then the minimiser $\mathbf{x}_k^{(a)}$ to

$$\mathcal{J}^{(3D)}(\mathbf{x}_k) = \left\langle B^{-1}\left(\mathbf{x}_k - \mathbf{x}_k^{(b)}\right), \mathbf{x}_k - \mathbf{x}_k^{(b)}\right\rangle_{\ell^2} + \left\langle R^{-1}(\mathbf{y}_k - H\mathbf{x}_k), \mathbf{y}_k - H\mathbf{x}_k\right\rangle_{\ell^2}, \tag{11}$$

is given by

$$\mathbf{x}_k^{(a)} = \mathbf{x}_k^{(b)} + \mathscr{K}\left(\mathbf{y}_k - H\mathbf{x}_k^{(b)}\right), \tag{12}$$

where

$$\mathscr{K} := BH'(HBH' + R)^{-1} \tag{13}$$

is the Kalman gain and H' is the adjoint with respect to the Euclidean inner product.

Theorem (Theorem 2.1 in *Marx and Potthast 2012*)

For the Euclidean inner product and the weighted norms

$$\langle \cdot, \cdot \rangle_{B^{-1}} := \langle \cdot, B^{-1} \cdot \rangle \text{ on } \mathbf{X} \text{ and } \langle \cdot, \cdot \rangle_{R^{-1}} := \langle \cdot, R^{-1} \cdot \rangle \text{ on } \mathbf{Y} \quad (14)$$

for self-adjoint, positive definite operators B and R, the Kalman gain $\mathscr{K}$ for 3DVar corresponds to the Tikhonov-Phillips inverse $\mathscr{R}_\alpha$ where its adjoint is given by

$$H^* = \alpha B H' R^{-1} \quad (15)$$

for $\alpha = 1$.

Outline

Full deterministic analysis update

From

$$\mathbf{x}_k^{(a)} = \mathcal{M}_{k-1}\left(\mathbf{x}_{k-1}^{(a)}\right) + \mathscr{R}_\alpha\left(\mathbf{y}_k - H\mathcal{M}_{k-1}\left(\mathbf{x}_{k-1}^{(a)}\right)\right), \tag{16}$$

we can derive

$$\mathbf{e}_k = N\left(\mathcal{M}_{k-1}\left(\mathbf{x}_{k-1}^{(a)}\right) - \mathcal{M}_{k-1}\left(\mathbf{x}_{k-1}^{(t)}\right)\right) + N\boldsymbol{\zeta}_k + \mathscr{R}_\alpha\boldsymbol{\eta}_k, \tag{17}$$

where $\mathbf{e}_k := \mathbf{x}_k^{(a)} - \mathbf{x}_k^{(t)}$, $N := I - \mathscr{R}_\alpha H$ and $\boldsymbol{\zeta}_k$ and $\boldsymbol{\eta}_k$ are the noise terms.

Full deterministic analysis update

As a first attempt,

$$\|\mathbf{e}_k\| \leq \|N\| \cdot \left\| \mathcal{M}_{k-1}\left(\mathbf{x}_{k-1}^{(a)}\right) - \mathcal{M}_{k-1}\left(\mathbf{x}_{k-1}^{(t)}\right) \right\| + \|N\| \, \upsilon + \|\mathscr{R}_\alpha\| \, \delta. \tag{18}$$

Assumption

The nonlinear mapping $\mathcal{M}_k : \mathbf{X} \to \mathbf{X}$ is Lipschitz continuous with a global Lipschitz constant such that given any $\mathbf{a}, \mathbf{b} \in \mathbf{X}$,

$$\|\mathcal{M}_k(\mathbf{a}) - \mathcal{M}_k(\mathbf{b})\| \leq K_k \cdot \|\mathbf{a} - \mathbf{b}\| \tag{19}$$

where $K_k \leq K$, the global Lipschitz constant for all time t_k, for $k \in \mathbb{N}_0$.

Therefore we obtain

$$e_k \leq \nu \cdot e_{k-1} + \|N\| \, \upsilon + \|\mathscr{R}_\alpha\| \, \delta, \tag{20}$$

where $\nu := K\|N\|$ given the global Lipschitz constant K.

Stability result

Theorem (*Moodey et al. 2013*)

For the Hilbert space $(\mathbf{X}, \|\cdot\|_{B^{-1}})$*, let the model error* $\boldsymbol{\zeta}_k$*,* $k \in \mathbb{N}_0$ *be bounded by* $\upsilon > 0$*. Let the observation error* $\boldsymbol{\eta}_k$*,* $k \in \mathbb{N}_0$ *be bounded by* $\delta > 0$*. If the nonlinear model operator* $\mathcal{M}_k : \mathbf{X} \to \mathbf{X}$ *is Lipschitz continuous and satisfies then the analysis error evolution* $e_k := \|\mathbf{e}_k\|$ *is estimated by*

$$e_k \leq \nu^k e_0 + \sum_{l=0}^{k-1} \nu^l \left(\|N\| \upsilon + \|\mathscr{R}_\alpha\| \delta \right), \tag{21}$$

for $k \in \mathbb{N}_0$*. If* $\nu < 1$ *then*

$$\limsup_{k \to \infty} e_k \leq \frac{\|N\| \upsilon + \|\mathscr{R}_\alpha\| \delta}{1 - \nu}. \tag{22}$$

Stability result

Lemma (*Potthast et al. 2012*)

Let $\mathbf{X}$ *and* $\mathbf{Y}$ *be Hilbert spaces and let* $H \in L(\mathbf{X}, \mathbf{Y})$ *be an injective compact linear operator, then the operator norm of the regularized reconstruction error operator is given by*

$$\|N\| = \|I - \mathscr{R}_\alpha H\| = 1. \tag{23}$$

This means to obtain $\nu := K\|N\| < 1$, the model operator $\mathcal{M}_k$ must be strictly damping, that is the global Lipschitz constant $K < 1$.

Stability result

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Question: Can we do better than this?

State space decomposition

- We use the singular system of the observation operator to split the state space. Let $(\mu_i, \varphi_i, \mathbf{g}_i)$ be the singular system of H.
- Let $P^{(1)}$ and $P^{(2)}$ be orthogonal projection operators, such that

$$P^{(1)} : \mathbf{X} \to \mathrm{span}\{\varphi_i, i \leq n\} \text{ and } P^{(2)} : \mathbf{X} \to \mathrm{span}\{\varphi_i, i > n\}, \tag{24}$$

for $i, n \in \mathbb{N}$.

- We define the following orthogonal subspaces

$$\mathbf{X}^{(1)} := \mathrm{span}\{\varphi_1, \ldots, \varphi_n\} \text{ and } \mathbf{X}^{(2)} := \mathrm{span}\{\varphi_{n+1}, \ldots, \varphi_\infty\}. \tag{25}$$

Nonlinear error evolution

Then we can expand our error evolution as follows,

$$\mathbf{e}_k = N\left(P^{(1)} + P^{(2)}\right)\left(\mathcal{M}_{k-1}\left(\mathbf{x}_{k-1}^{(a)}\right) - \mathcal{M}_{k-1}\left(\mathbf{x}_{k-1}^{(t)}\right)\right) + N\zeta_k + \mathscr{R}_\alpha \boldsymbol{\eta}_k \tag{26}$$

$$= N|_{\mathbf{X}^{(1)}}\left(\mathcal{M}_{k-1}^{(1)}\left(\mathbf{x}_{k-1}^{(a)}\right) - \mathcal{M}_{k-1}^{(1)}\left(\mathbf{x}_{k-1}^{(t)}\right)\right) + N\zeta_k + N|_{\mathbf{X}^{(2)}}\left(\mathcal{M}_{k-1}^{(2)}\left(\mathbf{x}_{k-1}^{(a)}\right) - \mathcal{M}_{k-1}^{(2)}\left(\mathbf{x}_{k-1}^{(t)}\right)\right) + \mathscr{R}_\alpha \boldsymbol{\eta}_k, \tag{27}$$

where

$$\mathcal{M}_k^{(1)}(\cdot) := P^{(1)} \circ \mathcal{M}_k(\cdot) \text{ and } \mathcal{M}_k^{(2)}(\cdot) := P^{(2)} \circ \mathcal{M}_k(\cdot). \tag{28}$$

Nonlinear error evolution

Taking norms and rearranging we have,

$$\|\mathbf{e}_k\| = \Big\| N|_{\mathbf{X}^{(1)}} \left(\mathcal{M}^{(1)}_{k-1}\left(\mathbf{x}^{(a)}_{k-1}\right) - \mathcal{M}^{(1)}_{k-1}\left(\mathbf{x}^{(t)}_{k-1}\right)\right) + N\boldsymbol{\zeta}_k + N|_{\mathbf{X}^{(2)}} \left(\mathcal{M}^{(2)}_{k-1}\left(\mathbf{x}^{(a)}_{k-1}\right) - \mathcal{M}^{(2)}_{k-1}\left(\mathbf{x}^{(t)}_{k-1}\right)\right) + \mathscr{R}_\alpha \boldsymbol{\eta}_k \Big\| \tag{29}$$

$$\leq \left(K^{(1)}_{k-1} \cdot \|N|_{\mathbf{X}^{(1)}}\| + K^{(2)}_{k-1} \cdot \|N|_{\mathbf{X}^{(2)}}\| \right) \cdot \left\| \mathbf{x}^{(a)}_{k-1} - \mathbf{x}^{(t)}_{k-1} \right\| + \|N\boldsymbol{\zeta}_k\| + \|\mathscr{R}_\alpha \boldsymbol{\eta}_k\| \tag{30}$$

where we assume Lipschitz continuity,

$$\left\| \mathcal{M}^{(j)}_{k-1}(\mathbf{x}^{(a)}_{k-1}) - \mathcal{M}^{(j)}_{k-1}(\mathbf{x}^{(t)}_{k-1}) \right\| \leq K^{(j)}_{k-1} \left\| \mathbf{x}^{(a)}_{k-1} - \mathbf{x}^{(t)}_{k-1} \right\| \tag{31}$$

for $j = 1, 2$, with restrictions according to the singular system of H.

Lemma (*Potthast et al. 2012*)

Let $(\mathbf{X}, \|\cdot\|_{B^{-1}})$ *be a Hilbert space with weighted norm and let* H *be an injective linear compact observation operator. Then, by choosing the regularization parameter* $\alpha > 0$ *sufficiently small we can find a parameter* $0 < \rho < 1$, *such that* $\|N|_{\mathbf{X}^{(1)}}\| \leq \rho < 1$.

Lemma (*Potthast et al. 2012*)

Let $(\mathbf{X}, \|\cdot\|_{B^{-1}})$ *be a Hilbert space with weighted norm and let* H *be an injective linear compact observation operator, then the operator norm of the regularized reconstruction error operator is given by*

$$\|N|_{\mathbf{X}^{(2)}}\| = 1. \tag{32}$$

Dissipation

- We require that $\nu < 1$. Applying our norm estimates we have that

$$\nu := K^{(1)} \cdot \|N|_{\mathbf{X}^{(1)}}\| + K^{(2)} \cdot \|N|_{\mathbf{X}^{(2)}}\| \leq K^{(1)} \cdot \rho + K^{(2)}. \quad (33)$$

- The nonlinear system $\mathcal{M}_k$ has to be damping in $\mathbf{X}^{(2)}$ for all time.

Definition

A nonlinear system $\mathcal{M}_k$, $k \in \mathbb{N}_0$, is *dissipative with respect to H* if it is Lipschitz continuous and damping with respect to higher spectral modes of H, in the sense that $\mathcal{M}_k^{(2)}$ satisfies

$$\left\| \mathcal{M}_k^{(2)}(\mathbf{a}) - \mathcal{M}_k^{(2)}(\mathbf{b}) \right\| \leq K_k^{(2)} \cdot \|\mathbf{a} - \mathbf{b}\| \quad (34)$$

$\forall\, \mathbf{a}, \mathbf{b} \in \mathbf{X}$, where $K_k^{(2)} \leq K^{(2)} < 1$ uniformly for $k \in \mathbb{N}_0$.

Final result

Under this assumption that $\mathcal{M}_k$ is dissipative with respect to H, we can choose the regularization parameter $\alpha > 0$ small enough, such that

$$\rho < \frac{1 - K^{(2)}}{K^{(1)}}, \tag{35}$$

to achieve a stable cycled scheme. We are now able to summarise this result in the following theorem.

Theorem (*Moodey et al. 2013*)

Let $(\mathbf{X}, \|\cdot\|_{B^{-1}})$ *be a Hilbert space with weighted norm. Let the nonlinear system* $\mathcal{M}_k : \mathbf{X} \to \mathbf{X}$ *be Lipschitz continuous and dissipative with respect to higher spectral modes of* H*. Then, for regularization parameter* $\alpha > 0$ *sufficiently small, we have* $\nu := K^{(1)}\|N|_{\mathbf{X}^{(1)}}\| + K^{(2)}\|N|_{\mathbf{X}^{(2)}}\| < 1$*. Then,*

$$\limsup_{k\to\infty} \|\mathbf{e}_k\| \le \frac{\|N\|\,\upsilon + \|\mathscr{R}_\alpha\|\,\delta}{1-\nu}. \tag{36}$$

Outline

Lorenz '63 equations

- The Lorenz '63 equations are as follows,

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -\sigma(x - y), \tag{37}$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \rho x - y - xz, \tag{38}$$

$$\frac{\mathrm{d}z}{\mathrm{d}t} = xy - \beta z, \tag{39}$$

  where typically σ, ρ and β are known as the Prandtl number, the Rayleigh number and a non-dimensional wave number respectively.
- We choose the classical parameters, $\sigma = 10$, $\rho = 28$ and $\beta = 8/3$ and discretise the system using a fourth order Runge-Kutta approximation.

Numerical experiment

- We set up a twin experiment and begin with an initial condition,

$$\left(x_0^{(t)}, y_0^{(t)}, z_0^{(t)}\right) = (-5.8696, -6.7824, 22.3356), \qquad (40)$$

- We produce a run of the system until time $t = 100$ with a step-size $h = 0.01$, which we call a *truth run*.
- Now we create observations at every tenth time-step from the truth run by adding random normally distributed noise with zero mean and standard deviation $\sigma_{(o)} = \sqrt{2}/40$.
- The background state is calculated in the same way at initial time t_0 with zero mean and standard deviation $\sigma_{(b)} = 1/400$ such that,

$$\left(x_0^{(b)}, y_0^{(b)}, z_0^{(b)}\right) = (-5.8674, -6.7860, 22.3338). \qquad (41)$$

- Now we calculate

$$\mathbf{e}_k = (I - \mathscr{R}_\alpha H)\left(\mathcal{M}_{k-1}\left(\mathbf{x}_{k-1}^{(a)}\right) - \mathcal{M}_{k-1}\left(\mathbf{x}_{k-1}^{(t)}\right)\right) + \mathscr{R}_\alpha \boldsymbol{\eta}_k. \qquad (42)$$

- We calculate a sampled background error covariance between the background state and true state over the whole trajectory such that

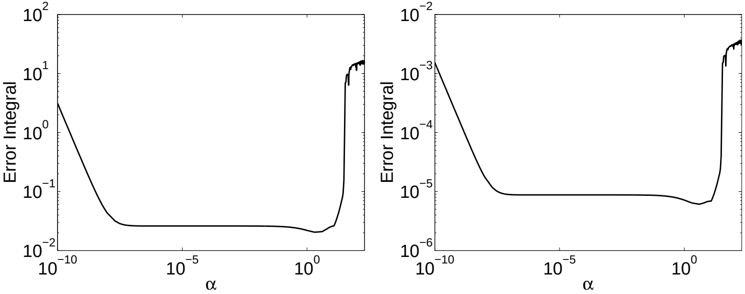
$$B = \begin{pmatrix} 117.6325 & 117.6374 & -2.3513 \\ 117.6374 & 152.6906 & -2.0838 \\ -2.3513 & -2.0838 & 110.8491 \end{pmatrix}. \quad (43)$$

- We simulate the consequence of an ill-posed observation operator H with a random 3×3 matrix with its last singular value $\mu_3 = 10^{-8}$ such that

$$H = \begin{pmatrix} 0.4267 & 0.5220 & 0.5059 \\ 0.8384 & -0.7453 & 1.6690 \\ 0.4105 & 1.6187 & 0.0610 \end{pmatrix}. \quad (44)$$

Therefore, H is severely ill-conditioned with a condition number, $\kappa = 2.1051 \times 10^8$.

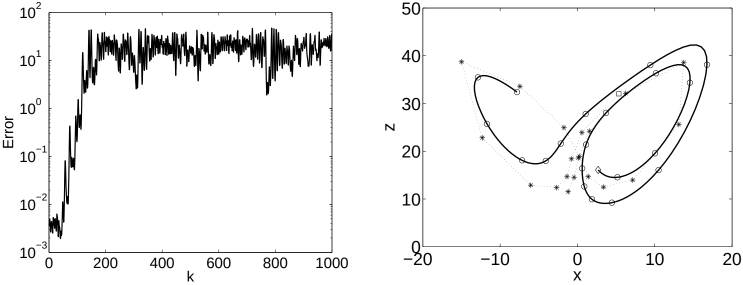
Numerical experiment



(a) (b)

Figure: (a) ℓ^2 norm of the analysis error $\|\mathbf{e}_k\|_{\ell^2}$ integrated for all assimilation time t_k for $k = 1, \ldots, 1000$, varying the regularization parameter, α. (b) Weighted norm of the analysis error $\|\mathbf{e}_k\|_{B^{-1}}$ integrated for all assimilation time t_k for $k = 1, \ldots, 1000$, varying the regularization parameter, α.

Numerical experiment



(a) (b)

Figure: (a) ℓ^2 norm of the analysis error $\|\mathbf{e}_k\|_{\ell^2}$ as the scheme is cycled for index k, with regularization parameter $\alpha = 200$, which corresponds to 3DVar. (b) Trajectories in state space for $t_{200:220}$.

Numerical experiment

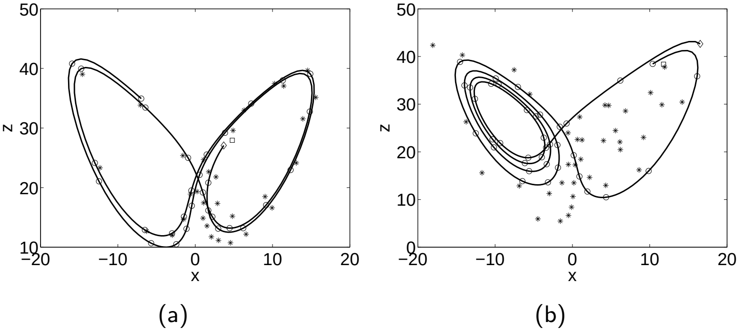
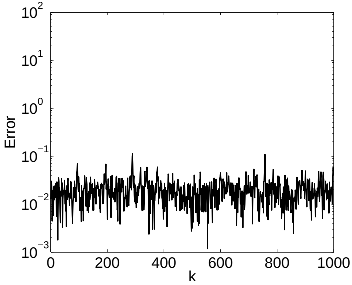
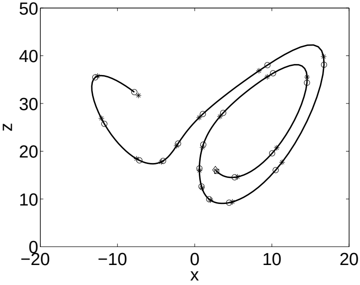


Figure: (a) and (b) Trajectories in state space for $t_{774:805}$ and $t_{858:895}$ respectively.

Numerical experiment



(a)



(b)

Figure: (a) ℓ^2 norm of the analysis error $\|\mathbf{e}_k\|_{\ell^2}$ as the scheme is cycled for the index k, with regularization parameter $\alpha = 2$, an inflation in the background variance of 100%. (b) Trajectories in state space for $t_{200:220}$.

Numerical experiment

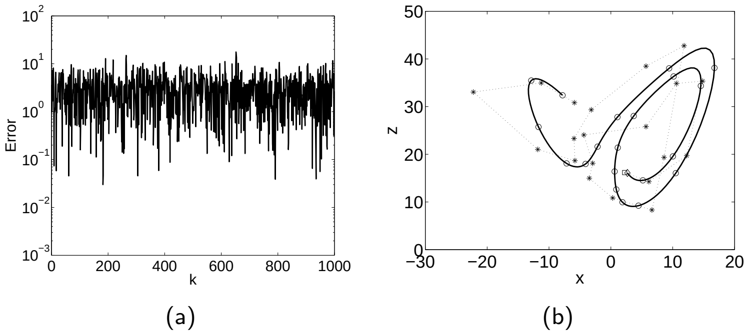
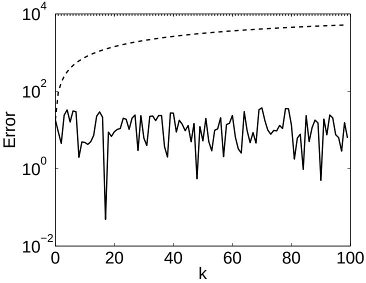
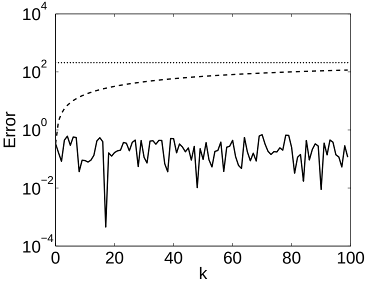


Figure: (a) ℓ^2 norm of the analysis error $\|\mathbf{e}_k\|_{\ell^2}$ as the scheme is cycled for the index k, with regularization parameter $\alpha = 10^{-10}$, an inflation in the background variance of $2 \times 10^{12}\%$. (b) Trajectories in state space for $t_{200:220}$.

Numerical experiment



(a)



(b)

Figure: (a) ℓ^2 norm of the analysis error, $\|\mathbf{e}_k\|_{\ell^2}$ as the scheme is cycled for the index k with regularization parameter, $\alpha = 10^{-6}$. (b) Weighted norm of the analysis error, $\|\mathbf{e}_k\|_{B^{-1}}$ as the scheme is cycled for the index k with regularization parameter, $\alpha = 10^{-6}$. Solid line: Nonlinear analysis error. Dashed line: Linear bound. Dotted line: Asymptotic limit.

Outline

Conclusion and future work

Conclusion

- We have developed new stability results for cycled data assimilation schemes with an ill-posed observation operator.
- Under weighted norms, the choice of the α was crucial to keep in the stable range.
- Dissipation in the nonlinear model dynamics is necessary for our stability result to hold.

Future work

- Investigate the analysis error in data assimilation scheme that employ an update in the background error covariance.

References

- R Kress. *Linear integral equations*. Applied mathematical sciences. Springer, 1999.
- B Marx and R W E Potthast. *On instabilities in data assimilation algorithms*, GEM, 3, 253-278, 2012.
- M Z Nashed. *Operator-theoretic and computational approaches to ill-posed problems with applications to antenna theory*, IEEE Transactions on Antennas and Propagation, 29, 220-231, 1981.
- M Z Nashed. *A new approach to classification and regularization of ill-posed operator equations*, Inverse and ill-posed problems, 53-75, 1987.
- B P Rynne and M A Youngson. *Linear functional analysis*. Springer, 2007.

- A J F Moodey, A S Lawless, R W E Potthast and P J van Leeuwen *Nonlinear error dynamics for cycled data assimilation methods*, Inverse Problems, 29, 025002, 2013.
- R W E Potthast, A J F Moodey, A S Lawless and P J van Leeuwen *On error dynamics and instability in data assimilation*, Submitted for publication, preprint available: MPS-2012-05 at http://www.reading.ac.uk/maths-and-stats/research/maths-preprints.aspx.