ACCURACY AND WELL-POSEDNESS OF GAUSSIAN FILTERS FOR DATA ASSIMILATION

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Outline

References

1 K.J.H.Law and A.M.Stuart. "Evaluating Data Assimilation Algorithms". Monthly Weather Review, **140**(2012), 3757–3782. arxiv.org/abs/1107.4118

2 C. Brett, A. Lam, K.J.H. Law, D. McCormick, M. Scott and A.M. Stuart. "Accuracy and Stability of Filters for the Dissipative PDEs”. PhysicaD, **245**(2013), 34–45. arxiv.org/abs/1110.2527

3 D. Blömker, K. Law, A.M. Stuart and K.Zygalakis. "Accuracy and stability of the continuous-time 3DVAR filter for the Navier-Stokes equation.” Nonlinearity, To appear. arxiv.org/abs/1210.1594

4 “Well-posedness of ensemble Kalman filters”
D.T.B Kelly and A.M. Stuart, In preparation.

5 K.J.H. Law, A. Shukla and A.M. Stuart, “Analysis of the 3DVAR Filter for the Partially Observed Lorenz ’63 Model.” Discrete and Continuous Dynamical Systems A, To Appear. arxiv.org/abs/1212.4923

Outline

The Filtering Problem

Partially Observed Dynamics

Let Ψ denote the time h flow of a dynamical system with uncertain initial condition:

$$v_{n+1} = \Psi(v_n), \quad v_0 \sim N(z_0, C_0)$$

with linear noisy observations

$$y_{n+1} = Hv_{n+1} + \zeta_{n+1}, \quad \zeta_n \sim N(0, \Gamma).$$

State Estimation

Try to estimate $\mathbb{P}\big(v_n|\{y_j\}_{j=1}^n\big)$. This probablity measure quantifies uncertainty.

Examples: Forward Model

Lorenz '63

$$\begin{aligned} \dot{v}^{(1)} &= \alpha\big(v^{(2)} - v^{(1)}\big), \\ \dot{v}^{(2)} &= -\alpha v^{(1)} - v^{(2)} - v^{(1)} v^{(3)}, \\ \dot{v}^{(3)} &= v^{(1)} v^{(2)} - b v^{(3)} - b\big(r + \alpha\big). \end{aligned}$$

Lorenz '96

$$\dot{v}^{(k)} = v^{(k-1)}\Big(v^{(k+1)} - v^{(k-2)}\Big) - v^{(k)} + f, \quad k = 1, \cdots, K$$

$$v^{(0)} = v^{(K)}, \; v^{(-1)} = v^{(K-1)}, \; v^{(K+1)} = v^{(1)}.$$

2D Navier-Stokes as ODE on $\mathcal{H}$:

$$\mathcal{H} := \dot{L}^2_{\mathrm{div}}(\mathbb{T}^2). \qquad \mathcal{V} := \dot{H}^1_{\mathrm{div}}(\mathbb{T}^2).$$

Examples: Observation Operator

Lorenz '63

Observe only $v^{(1)}$ so that $H = (1, 0, 0)^T$.

Lorenz '96

Observe $2/3$ of the variables (regular pattern, not random).

2D Navier-Stokes as ODE on $\mathcal{H}$:

Observe Fourier modes with wave number $k : |k| \le k_{\max}$.

General Setting

For $(\mathcal{V}, \|\cdot\|)$ continuously embedded in $\big(\mathcal{H}, \langle\cdot,\cdot\rangle, |\cdot|\big)$:

The Model

$$\frac{dv}{dt} + Av + B(v, v) = f$$

Assumptions 1

There is $\lambda > 0$ such that, for all $w \in \mathcal{V}$,

$$\langle Aw, w\rangle \geq \lambda\|w\|^2, \quad \langle B(w, w), w\rangle = 0.$$

Assumptions 2

The symmetric bilinear form B satisfies, for all $w_i \in \mathcal{V}$,

$$\langle B(w_1, w_2), w_2\rangle \leq K\|w_1\|\|w_2\||w_2|, \quad \langle B(w_1, w_2), w_3\rangle \leq K\|w_1\|\|w_2\|\|w_3\|.$$

Approximate Gaussian Filters

Prediction Step

$$\widehat{z}_{n+1} = \Psi(z_n).$$

Analysis Step

$$z_{n+1} = \mathrm{argmin}_z \Big(|C_{n+1}^{-\frac{1}{2}}(z - \widehat{z}_{n+1})|^2 + |\Gamma^{-1/2}(y_{n+1} - Hz)|^2 \Big)$$

Design Parameters

The operators C_{n+1} characterize model uncertainty and are design parameters.

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3DVAR: Approximate Gaussian Filter

Fixed Model Covariance

$$C_{n+1} = C$$

Kalman Mean Update

$$z_{n+1} = (I - KH)\Psi(z_n) + Ky_{n+1}.$$

Kalman Covariance Update

$$K = CH^*(HCH^* + \Gamma)^{-1}.$$

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Accuracy Theorem

- Recall $v_n = v(nh)$. Truth.
- Define $\epsilon^2 := \mathbb{E}\|\zeta_n\|^2$. Observational noise variance.
- Define η^2. Ratio of observational to model variance.

Assumptions 3

Let h be sufficiently small and, for NSE, $k_{\max}$ sufficiently large.

Theorem 1 ([2,5])

Under Assumptions 1,2, and 3 there exists $r \in (0,1)$ and $\eta_c, \mathrm{k} \in (0,\infty)$ such that, for all $\eta < \eta_c$,

$$\mathbb{E}\|z_j - v_j\|^2 \le r^j \|z_0 - v_0\|^2 + \frac{\mathrm{k}}{h}\epsilon^2$$

Proof of Accuracy/Stability Theorem

Rewrite Truth Update

$$
\begin{aligned}
v_{n+1} &= \Psi(v_n) \\
v_{n+1} &= (I - KH)\Psi(v_n) + KH\Psi(v_n)
\end{aligned}
$$

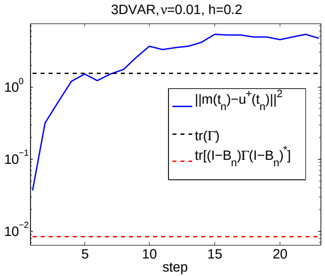
Use Data in Filter

$$
\begin{aligned}
z_{n+1} &= (I - KH)\Psi(z_n) + Ky_{n+1} \\
z_{n+1} &= (I - KH)\Psi(z_n) + KH\Psi(v_n) + K\xi_n.
\end{aligned}
$$

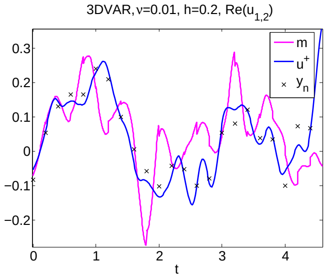
Subtract

$$
z_{n+1} - v_{n+1} = (I - KH)\Big(\Psi(z_n) - \Psi(v_n)\Big) + K\xi_n.
$$

Inaccurate

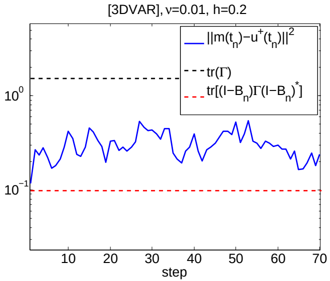


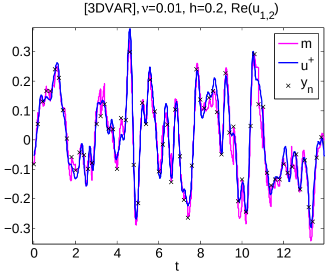
3DVAR, ν=0.01, h=0.2
10^0
10^{-1}
10^{-2}
5
10
15
20
step
$||m(t_n)-u^+(t_n)||^2$
tr(Γ)
$tr[(I-B_n)\Gamma(I-B_n)^*]$



3DVAR, ν=0.01, h=0.2, $Re(u_{1,2})$
0.3
0.2
0.1
0
−0.1
−0.2
0
1
2
3
4
t
m
u^+
y_n

Accurate: Variance Inflation, Decrease η





Outline

Parameter Scalings

Rescale

- $C = \omega \sigma_0^2 A^{-2\zeta}$;
- $\Gamma = \frac{1}{h} \sigma_0^2 A^{-2\beta}$;
- $y_{j+1} := \left(\frac{\xi_{j+1} - \xi_j}{h}\right)$ and ξ_j is now viewed as the data;
- $\zeta_j \sim \frac{\sigma_0}{\sqrt{h}} N(0, A^{-2\beta})$.

SPDE Limit

Limiting S(P)DE

$$\frac{dz}{dt} + \nu Az + B(z,z) + \omega H^* A^{-2\alpha}\Big(Hz - \frac{d\xi}{dt}\Big) = f.$$

Data

$$\frac{d\xi}{dt} = Hv + \sigma_0 A^{-\beta} H \frac{dW}{dt}.$$

Combining

$$\frac{dz}{dt} + \nu Az + B(z,z) + \omega A^{-2\alpha} H\Big(z - v\Big) = f + \sigma_0 \omega A^{-2\alpha-\beta} H \frac{dW}{dt}.$$

Accuracy Theorem

- Recall $v(t)$. Truth.
- Define $\epsilon^2 := \omega^2\sigma_0^2$.

Assumptions 4

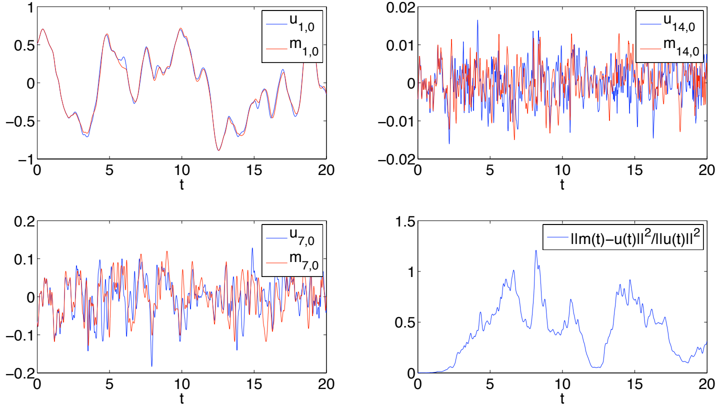
(For NSE only) $k_{\max}$ sufficiently large.

Theorem 2 ([3,5])

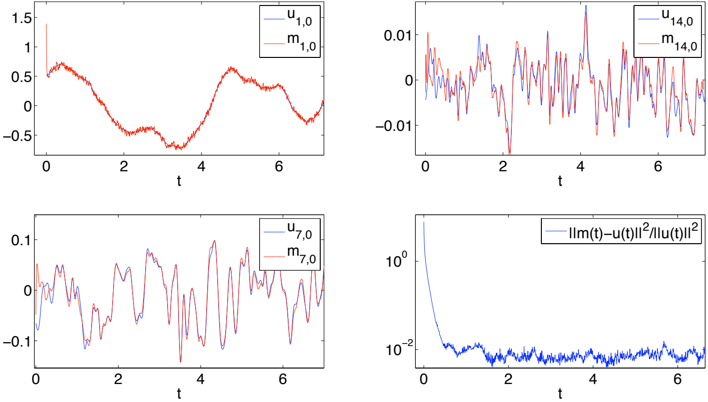
Under Assumptions 1,2 and 4 there exists $\omega_c > 0$ and $\gamma_0, K \in (0, \infty)$ such that, for all $\omega > \omega_c$,

$$\mathbb{E}|z(t) - v(t)|^2 \leq \exp(-\gamma_0 t)|z(0) - v(0)|^2 + c\epsilon^2$$

SPDE Inaccurate



SPDE Accurate: Decrease σ_0



Outline

The Ensemble Kalman Filter

Prediction Step

$$\widehat{z}^j_{n+1} = \Psi(z^j_n), \quad j \in \{1, \cdots, J\}.$$

Estimate model uncertainty:

$$\overline{z}_{n+1} = \frac{1}{J}\sum_{j=1}^{J} \widehat{z}^j_{n+1}$$
$$C_{n+1} = \frac{1}{J}\sum_{j=1}^{J} \widehat{z}^j_{n+1}(\widehat{z}^j_{n+1})^T - \overline{z}_{n+1}\overline{z}^T_{n+1}$$

Analysis Step

$$S_{n+1} = HC_{n+1}H^T + \Gamma, \qquad K_{n+1} = C_{n+1}H^T S^{-1}_{n+1}$$
$$z^j_{n+1} = (I - K_{n+1}H)\widehat{z}^j_{n+1} + K_{n+1}y^j_{n+1}, \quad j \in \{1, \cdots, J\}.$$

Perturbed Observations Data

$$y^j_n = y_n + \zeta^j_n, \qquad \zeta^j_n \sim N(0, \Gamma)$$

Why Use EnKF?

Relative Errors in Filters for Chaotic NSE [1]

method	*error*
3DVAR($h = 0.02$)	0.063289
EnKF($h = 0.02$)	0.0523566
3DVAR($h = 0.1$)	0.203165
EnKF($h = 0.1$)	0.109402
3DVAR($h = 0.2$)	0.300853
EnKF($h = 0.2$)	0.113806

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Discrete Time Filter: Well-Posedness

Recall v_n truth.

Assumptions 5

$$\Gamma = \gamma^2 I, H = I.$$

Theorem 3 ([4])

Under Assumptions 1,2 and 5 there are constants β, K independent of n such that

$$\mathbb{E}|z_n^j - v_n|^2 \leq \exp(2\beta nh)\mathbb{E}|z_0^j - z_0|^2 + 2K\gamma^2\Big(\frac{\exp(2\beta nh) - 1}{\exp(2\beta h) - 1}\Big).$$

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Continuous Time Limit

Scalings

$$\Gamma = \frac{1}{h}\Gamma_0, \quad z_n = z(nh), \quad z_n^j = z^j(nh), \quad h \ll 1.$$

Limiting S(P)DEs

$$\frac{dz^j}{dt} + Az^j + B(z^j, z^j) = f + CH^*\Gamma_0^{-1}\Big(\frac{d\xi^j}{dt} - Hz^j\Big),$$

$$\frac{d\xi^j}{dt} = Hv + \Gamma_0^{\frac{1}{2}} H \frac{dW^j}{dt}.$$

Coupling

Coupled through the empirical covariance C

Continuous Time Limit: Well-Posedness

Recall $v(t)$ truth.

Assumptions 6

$$\Gamma_0 = \gamma^2 I, H = I.$$

Theorem 4 ([4])

Under Assumptions 1,2 and 6 there are constants β, K independent of t such that

$$\mathbb{E}\sum_{j=1}^{J} |z^j(t) - v(t)|^2 \leq \exp(2\beta t)\mathbb{E}\sum_{j=1}^{J} |z^j(0) - z(0)|^2 + 2K\left(\frac{\exp(2\beta t) - 1}{\exp(2\beta) - 1}\right).$$

Outline

Summary

- Approximate Gaussian filters solve a minimization problem, sequentially, to compute a model/data compromise.
- Quadratic dissipative systems are studied to get insight into optimal paramater choices.
- The 3DVAR method is provably accurate, in both discrete and continuous time settings, if enough unstable modes are observed and data is trusted enough (variance inflation).
- The EnKF is provably well-posed, in both discrete and continuous time settings, in fully observed case.
- Filter instability reported in the literature probably involves interaction with numerical instability.
- Significant hurdles in analysis to push rigorous study of EnKF further.

Outline

References

1. K.J.H.Law and A.M.Stuart. "Evaluating Data Assimilation Algorithms". Monthly Weather Review, **140**(2012), 3757–3782. `arxiv.org/abs/1107.4118`
2. C. Brett, A. Lam, K.J.H. Law, D. McCormick, M. Scott and A.M. Stuart. "Accuracy and Stability of Filters for the Dissipative PDEs". PhysicaD, **245**(2013), 34–45. `arxiv.org/abs/1110.2527`
3. D. Blömker, K. Law, A.M. Stuart and K.Zygalakis. "Accuracy and stability of the continuous-time 3DVAR filter for the Navier-Stokes equation." Nonlinearity, To Appear. `arxiv.org/abs/1210.1594`
4. “Well-posedness of ensemble Kalman filters” DB Kelly and AM Stuart, In preparation.
5. K.J.H. Law, A. Shukla and A.M. Stuart, “Analysis of the 3DVAR Filter for the Partially Observed Lorenz ’63 Model.” Discrete and Continuous Dynamical Systems A, To Appear. `arxiv.org/abs/1212.4923`