Smith Cloud Scheme

Variables
T_L liquid water temperature
q_W total water (all phases)
q_C grid box mean cloud condensate. This quantity must be non-negative.
Q_C mode of cloud distribution. This quantitiy can be positive, zero or negative.
s local deviation from Q_C
q'_C local cloud water content (note - although this variable is primed, it is not a perturbation)
C cloud fraction

Cloud conserved variables

$$T_L = T - Lq_C / c_p \tag{1}$$

$$q_W = q + q_C \tag{2}$$

Prognostic equations

$$\frac{\mathrm{d}\, q_C}{\mathrm{d}\, t} = (\dot{q}_C)_A + (\dot{q}_C)_D + (\dot{q}_C)_{TM} + (\dot{q}_C)_S + (\dot{q}_C)_P + (\dot{q}_C)_{CV} \tag{3}$$

$$\frac{\mathrm{d}\, q}{\mathrm{d}\, t} = (\dot{q})_A + (\dot{q})_D + (\dot{q})_{TM} + (\dot{q})_S + (\dot{q})_P + (\dot{q})_{CV} \tag{4}$$

$$\frac{\mathrm{d}\, T}{\mathrm{d}\, t} = (\dot{T})_A + (\dot{T})_D + (\dot{T})_M + (\dot{T})_R + (\dot{T})_{TM} + (\dot{T})_S + (\dot{T})_E + (\dot{T})_{MF} + (\dot{T})_{CV} \tag{5}$$

Notes for cloud conserved variables

$$(\dot{q}_W)_S = (\dot{q}_C)_S + (\dot{q})_S \equiv 0 \tag{6}$$

$$(\dot{T}_L)_S = (\dot{T})_S - L(\dot{q}_C)_S \equiv 0 \tag{7}$$

Equation (2.19) - the grid box mean cloud water

$$Q_C = q_W - q_{SAT}(T) \tag{8}$$

Assume that T is not know, only q_W and T_W are known. From (1) and (8)

$$Q_C = q_W - q_{SAT}(T_L + LQ_C / c_p) \tag{9}$$

Solve a linearised version of this equation for Q_C

$$Q_C \approx q_W - q_{SAT}(T_L) - \left.\frac{\mathrm{d}\, q_{SAT}}{\mathrm{d}\, T}\right|_{T_L} \frac{LQ_C}{c_p}$$

$$\therefore Q_C \approx a_L(q_W - q_{SAT}(T_L)) \tag{10}$$

$$a_L = \left(1 + \left.\frac{\mathrm{d}\, q_{SAT}}{\mathrm{d}\, T}\right|_{T_L} \frac{L}{c_p}\right)^{-1} \tag{11}$$

Equation (2.20) - the perturbation to the grid box mean cloud water
Perturb the known variables (q_W and T_W) by q'_W and T'_W. What is the perturbation in Q_C, s?

$$s = Q'_C = a_L(q_W + q'_W - q_{SAT}(T_L + T'_L)) - a_L(q_W - q_{SAT}(T_L))$$

$$= a_L q'_W - a_L q_{SAT}(T_L + T'_L) + a_L q_{SAT}(T_L)$$

$$\approx\ a_L q'_W - a_L q_{SAT}(T_L) - a_L \left.\frac{\mathrm{d}\,q_{SAT}}{\mathrm{d}\,T}\right|_{T_L} T'_L + a_L q_{SAT}(T_L)$$

$$\approx\ a_L q'_W - a_L \left.\frac{\mathrm{d}\,q_{SAT}}{\mathrm{d}\,T}\right|_{T_L} T'_L$$

$$\approx\ a_L \left(q'_W - \alpha_L T'_L\right) \tag{12}$$

$$\alpha_L \equiv \left.\frac{\mathrm{d}\,q_{SAT}}{\mathrm{d}\,T}\right|_{T_L} \tag{13}$$

Equation (2.21) - the standard deviation of s

$$\sigma_s = \left(\overline{(s - \bar{s})^2}\right)^{1/2} = \left(\overline{(a_L(q'_W - \alpha_L T'_L))^2}\right)^{1/2}$$

$$= a_L \left(\overline{q_W'^2} + \alpha_L^2 \overline{T_L'^2} - 2\alpha_L \overline{q'_W T'_L}\right)^{1/2} \tag{14}$$

Equation (2.22) - local cloud water content

$$q'_C = Q_C + s \tag{15}$$

as long as q'_C is non-negative. Set to zero if negative - this leads to Eq. (2.22).

Equation (2.23) - mean cloud fraction

This is the fraction of the volume of the grid box that has cloud. This could be defined via the volume integral

$$C = \frac{\int \mathrm{d}\,V\,\delta(q'_C > 0)}{\int \mathrm{d}\,V} \qquad \delta(q'_C > 0) \equiv \begin{cases} 0 & \text{for } q'_C = 0 \\ 1 & \text{for } q'_C > 0 \end{cases} \tag{16}$$

Let the normalised distribution function for cloud be $G(s)$. Then

$$C = \int_{-\infty}^{\infty} \mathrm{d}\,s\,G(s)\,\delta(q'_C > 0)$$

where the δ-function is present to allow only those s that have non-zero q'_C. This can be achieved by adjusting the integration limits

$$C = \int_{-Q_C}^{\infty} \mathrm{d}\,s\,G(s) \tag{17}$$

Equation (2.24) - the grid box mean cloud water content

$$q_C = \int_{-Q_C}^{\infty} \mathrm{d}\,s\,q'_C G(s) = \int_{-Q_C}^{\infty} \mathrm{d}\,s\,\left(Q_C + s\right) G(s) \tag{18}$$

N.B. What is the difference between Q_C and q_C?

What is the standard deviation of G?

The functional form of G

$$G(s) = \begin{cases} s/(6\sigma_s^2) + 1/(\sqrt{6}\sigma_s) & -\sqrt{6}\sigma_s \leqslant s \leqslant 0 \\ -s/(6\sigma_s^2) + 1/(\sqrt{6}\sigma_s) & 0 < s \leqslant \sqrt{6}\sigma_s \\ 0 & \text{otherwise} \end{cases} \tag{19}$$

$$\sigma_s = \left(\overline{(s - \bar{s})^2}\right)^{1/2} = \left(\int_{s=-\infty}^{\infty} \mathrm{d}s\, s^2 G(s)\right)^{1/2}$$

$$= \left(\int_{s=-\sqrt{6}\sigma_s}^{0} \mathrm{d}s\, s^2 \left(\frac{s}{6\sigma_s^2} + \frac{1}{\sqrt{6}\sigma_s}\right) + \int_{s=0}^{\sqrt{6}\sigma_s} \mathrm{d}s\, s^2 \left(-\frac{s}{6\sigma_s^2} + \frac{1}{\sqrt{6}\sigma_s}\right)\right)^{1/2}$$

$$= \left(\left[\frac{s^4}{24\sigma_s^2} + \frac{s^3}{3\sqrt{6}\sigma_s}\right]_{-\sqrt{6}\sigma_s}^{0} + \left[-\frac{s^4}{24\sigma_s^2} + \frac{s^3}{3\sqrt{6}\sigma_s}\right]_{0}^{\sqrt{6}\sigma_s}\right)^{1/2}$$

$$= \left(-\left(\frac{36\sigma_s^4}{24\sigma_s^2} - \frac{6\sqrt{6}\sigma_s^3}{3\sqrt{6}\sigma_s}\right) + \left(-\frac{36\sigma_s^4}{24\sigma_s^2} + \frac{6\sqrt{6}\sigma_s^3}{3\sqrt{6}\sigma_s}\right)\right)^{1/2}$$

$$= \left(4\sigma_s^2 - 3\sigma_s^2\right)^{1/2} = \sigma_s \qquad (20)$$

This confirms the form of $G(s)$ in Fig. 1 of the paper.

Equation (2.25) - the critical relative humidity

Cloud just forms when q_C' is just zero everywhere. Insert $q_C' = 0$ and $s = \sqrt{6}\sigma_s$ into (15). This gives $Q_C = -\sqrt{6}\sigma_s$. Since there is no condenstate, all water is vapour, $q = q_W$. The relative humidity under these conditions is RH_C. From (10)

$$-\sqrt{6}\sigma_s = a_L(q - q_{SAT}(T_L))$$

$$\frac{\sqrt{6}\sigma_s}{q_{SAT}(T_L)} = a_L\left(1 - \frac{q}{q_{SAT}(T_L)}\right)$$

$$= a_L(1 - RH_C)$$

$$RH_C = 1 - \frac{\sqrt{6}\sigma_s}{a_L q_{SAT}(T_L)} \qquad (21)$$

Equations (2.26) to (2.28) - to make the critical relative humidity independent of temperature

As it stands, (21) is dependent upon temperature (through q_{SAT} and a_L). We may choose σ_s such that this is not the case. Choose

$$\sigma_s = A a_L q_{SAT}(T_L) f(q_W / q_{SAT}(T_L)) \qquad (22)$$

where $q_W / q_{SAT}(T_L) = RH_C$ at this point, A is a constant and f is an arbitrary function. A follows from (21) and (22)

$$RH_C = 1 - \sqrt{6} A f(q_W / q_{SAT}(T_L))$$

$$A = \frac{1 - RH_C}{\sqrt{6} f(q_W / q_{SAT}(T_L))} \qquad (23)$$

For simplicity, the paper chooses $f = 1$ leading (22) to

$$\sigma_s = \frac{a_L q_{SAT}(T_L)(1 - RH_C)}{\sqrt{6}} \qquad (24)$$