Example Gaussians in two dimensions

January 23, 2023

Starting point

A Gaussian distribution has the following form

$$p(\mathbf{x}) = \frac{1}{\sqrt{(2\pi)^n \det(\mathbf{P})}} \exp\left[-\frac{1}{2}\left(\mathbf{x} - \langle\mathbf{x}\rangle\right)^{\mathrm{T}} \mathbf{P}^{-1} \left(\mathbf{x} - \langle\mathbf{x}\rangle\right)\right],$$

where $\mathbf{x}$ is the vector of data, $\langle\mathbf{x}\rangle$ is its mean, $\mathbf{P}$ is its error covariance, and n is the dimensionality of $\mathbf{x}$. These notes show some example Gaussian distributions for $n = 2$. Let $\mathbf{P}$ have the following form when $\mathbf{x} = \begin{pmatrix} x_1 & x_2 \end{pmatrix}^{\mathrm{T}}$:

$$\mathbf{P} = \begin{pmatrix} v_1 & \gamma\sqrt{v_1 v_2} \\ \gamma\sqrt{v_1 v_2} & v_2 \end{pmatrix},$$

where v_1 and v_2 are the variances of x_1 and x_2 respectively, and γ is the correlation between x_1 and x_2 ($-1 \leq \gamma \leq 1$). The determinant of $\mathbf{P}$ is

$$\det(\mathbf{P}) = v_v v_2 (1 - \gamma^2),$$

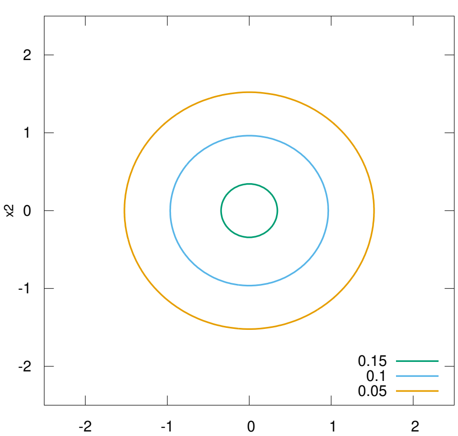
and the inverse of $\mathbf{P}$ is

$$\mathbf{P}^{-1} = \frac{1}{v_v v_2 (1 - \gamma^2)} \begin{pmatrix} v_2 & -\gamma\sqrt{v_1 v_2} \\ -\gamma\sqrt{v_1 v_2} & v_1 \end{pmatrix}.$$

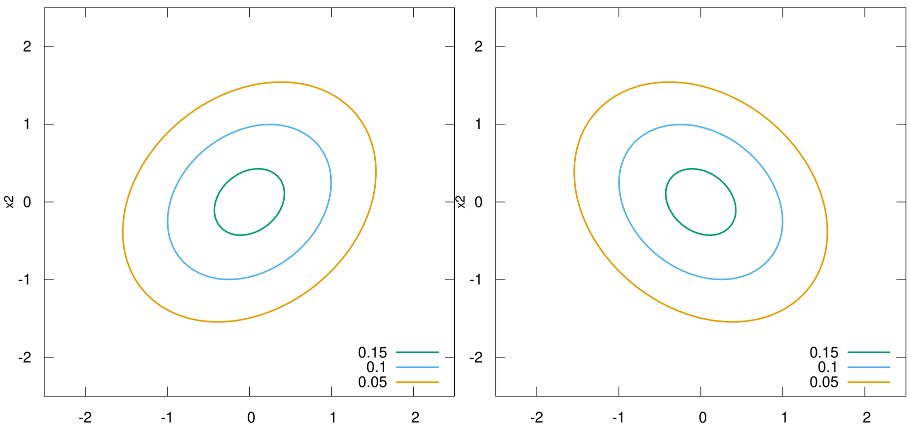
The determinant and the inverse both appear in the Gaussian form.

Example distributions

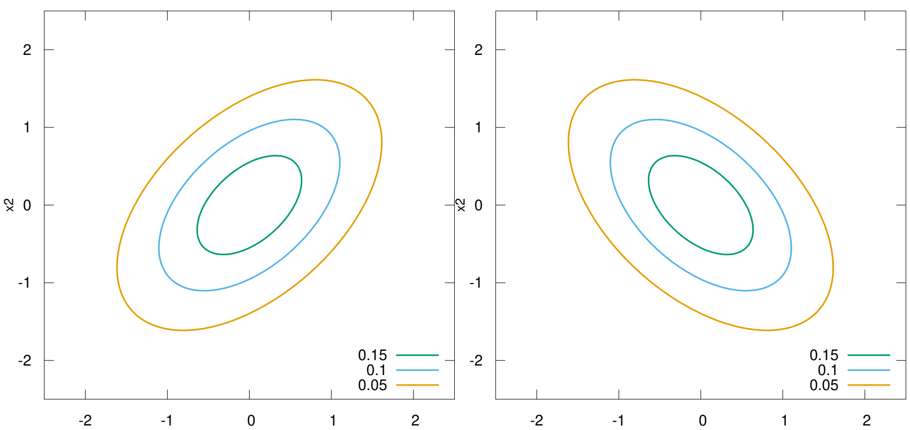
Form 1: $v_1 = 1$, $v_2 = 1$, $\gamma = 0$ (equal variances, no correlation)



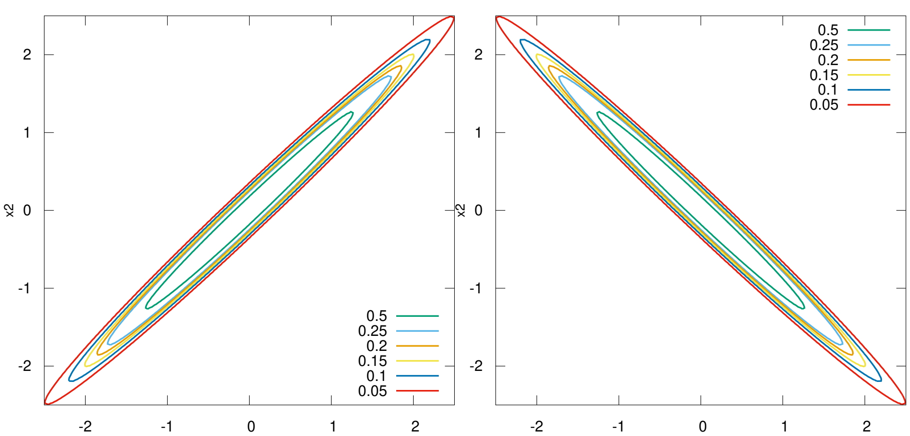
Form 2: $v_1 = 1,\ v_2 = 1,\ \gamma = \pm 0.25$ **(equal variances, weak positive/negative correlation)**



Form 3: $v_1 = 1,\ v_2 = 1,\ \gamma = \pm 0.5$ **(equal variances, moderate positive/negative correlation)**

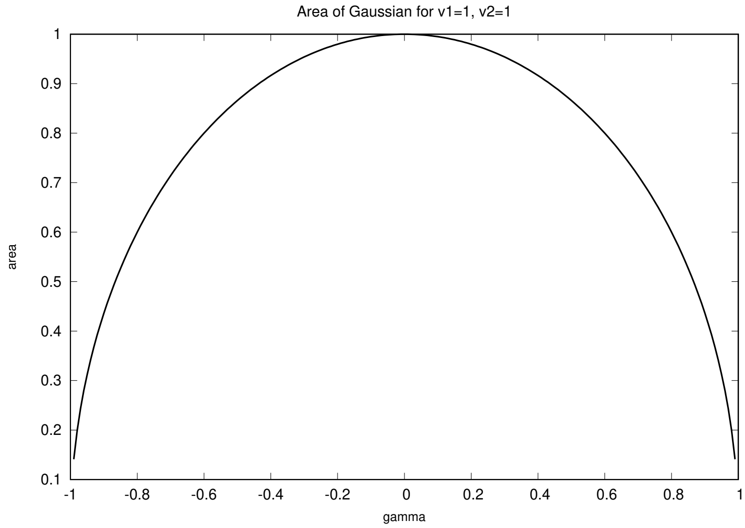


Form 4: $v_1 = 1,\ v_2 = 1,\ \gamma = \pm 0.99$ **(equal variances, high positive/negative correlation)**

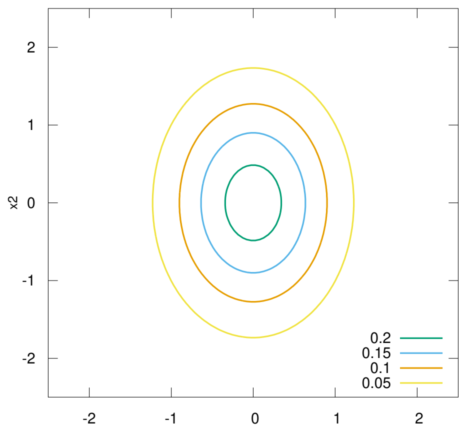


Summary of 'area' of Gaussian for $v_1 = 1,\ v_2 = 1$

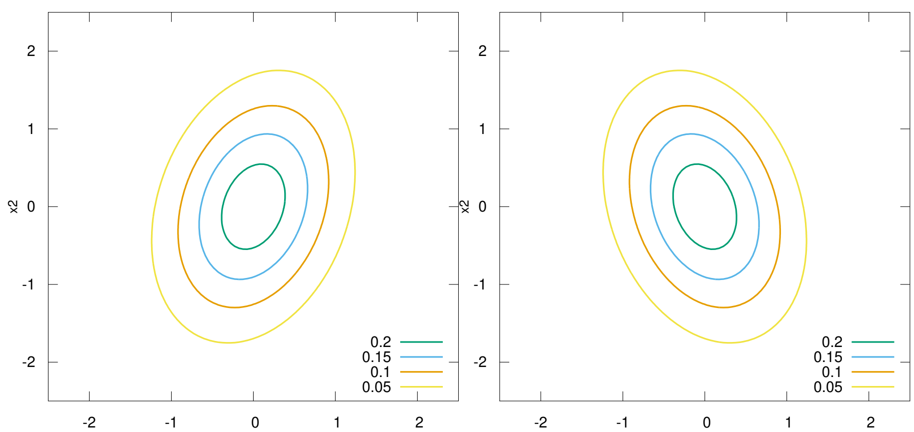
Plotted is $\sqrt{\lambda_1 \lambda_2}$ as a function of correlation, γ, where λ_i is the ith eigenvalue of the specified error covariance matrix. The area values plotted are just to show how the relative values change with γ.



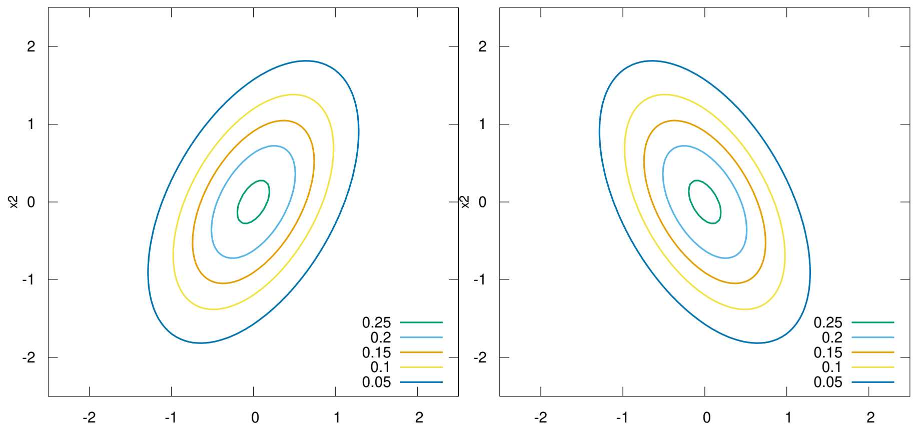
Form 5: $v_1 = 1$, $v_2 = 0.5$, $\gamma = 0$ **(unequal variances, no correlation)**



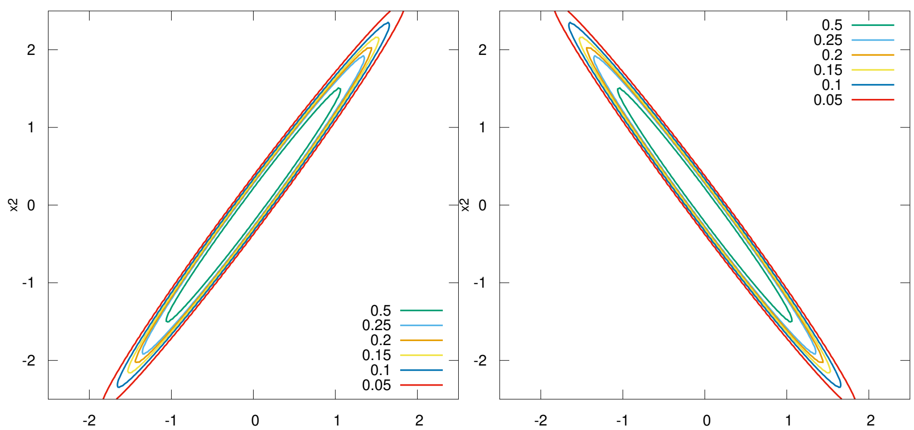
Form 6: $v_1 = 1$, $v_2 = 0.5$, $\gamma = \pm 0.25$ **(unequal variances, weak positive/negative correlation)**



Form 7: $v_1 = 1,\ v_2 = 0.5,\ \gamma = \pm 0.5$ **(unequal variances, moderate positive/negative correlation)**

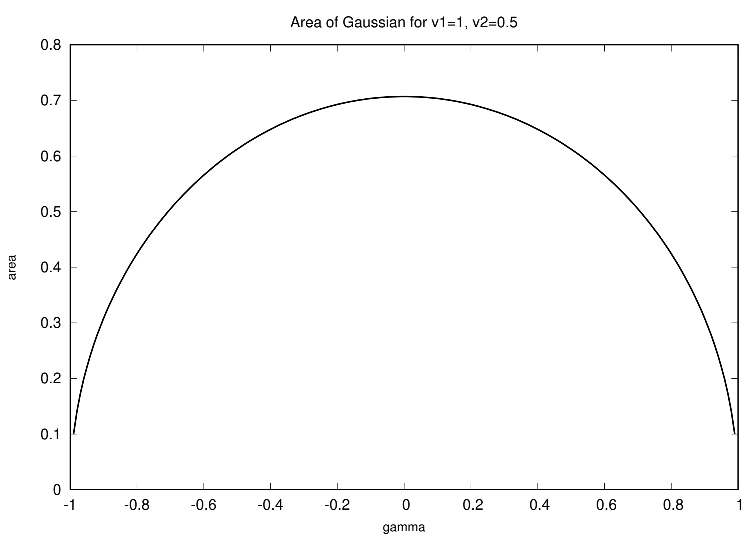


Form 8: $v_1 = 1,\ v_2 = 0.5,\ \gamma = \pm 0.99$ **(unequal variances, high positive/negative correlation)**



Summary of 'area' of Gaussian for $v_1 = 1,\ v_2 = 0.5$

Plotted is $\sqrt{\lambda_1 \lambda_2}$ as a function of correlation, γ, where λ_i is the ith eigenvalue of the specified error covariance matrix. The area values plotted are just to show how the relative values change with γ.



RNB