MSc Exam question (3d-Var) 2004. Ross Bannister

A three-dimensional variational (3d-Var.) cost function, J, is,

$$J = \tfrac{1}{2}(\vec{x}-\vec{x}_B)^T \mathbf{B}^{-1}(\vec{x}-\vec{x}_B) + \tfrac{1}{2}(\vec{y}-\vec{h}[\vec{x}])^T \mathbf{R}^{-1}(\vec{y}-\vec{h}[\vec{x}])$$

where $\vec{x}$ is the state vector describing the atmosphere, $\vec{x}_B$ is the first-guess or background state, $\mathbf{B}$ is the background error covariance matrix, $\vec{y}$ is the observation vector, $\vec{h}[\vec{x}]$ is the forward model, predicting the observations and $\mathbf{R}$ is the observation error covariance matrix. In this question the time evolution of the atmosphere is ignored.

(a) Describe the principles of variational data assimilation. Describe briefly and qualitatively:

- (i) how the analysis vector, $\vec{x}_A$ is found, [1 mark]
- (ii) the role played by each term, and [2 marks]
- (iii) the importance of the error covariance matrices. [1 mark]

(b) A model has a single grid box and carries two variables: pressure, p, and potential temperature, θ (defined below). A variational procedure is to be used to infer these variables from two uncorrelated temperature observations T_1 and T_2, each with observational error ΔT. Let $\vec{x}_B$ be the first guess of the model's state.

To define your notation, write down the following, with components, that represent:

- (i) the observations, $\vec{y}$, [1 mark]
- (ii) the model's state, $\vec{x}$, and [1 mark]
- (iii) the observation error covariance matrix, $\mathbf{R}$. [1 mark]

In order to compute the first iteration in 3d-Var.:

- (iii) Differentiate J with respect to each component of $\vec{x}$ and write the gradients in compact vector form using $\mathbf{H}$ as the Jacobian matrix, $\mathbf{H} = \partial\vec{h}/\partial\vec{x}$. [5 marks]
- (iv) How many matrix elements does $\mathbf{H}$ have? [1 mark]
- (v) Potential temperature is related to temperature and pressure by,

$$\theta = \left(\frac{p}{p_0}\right)^{-\kappa} T$$

where p_0 and κ are constants. Use this information to write all elements of the Jacobean. [4 marks]

- (vi) In this problem, how many elements has $\mathbf{B}$? [1 mark]
- (vii) Explain briefly how the gradient information from (biii) is used to increment the first guess model state in the first Var. iteration. [2 marks]

(c) According to the best linear unbiased estimator (BLUE), the analysis vector is,

$$\vec{x}_a = \vec{x}_B + \mathbf{BH}^T(\mathbf{HBH}^T + \mathbf{R})^{-1}$$

- (i) Generally speaking, under what circumstances does BLUE give the same analysis as the converged 3d-Var? [1 mark]
- (ii) Specify how 3d-Var. is superior to BLUE in numerical weather prediction. [4 marks]