MSc in Data Assimilation, Exam Question for 3d-Var (2008)
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A 3d-Var. cost function, J, has the following form in terms of the usual symbols

$$J = \tfrac{1}{2}(\vec{x} - \vec{x}_{\mathrm{B}})^{\mathrm{T}} \mathbf{B}^{-1} (\vec{x} - \vec{x}_{\mathrm{B}}) + \tfrac{1}{2}(\vec{y} - \vec{h}[\vec{x}])^{\mathrm{T}} \mathbf{R}^{-1} (\vec{y} - \vec{h}[\vec{x}]).$$

Let $\vec{x}$ consist of the quantities temperature (T), water vapour concentration (q), and pressure (p), each on a three-dimensional (longitude, latitude and height) grid covering the Earth's atmosphere. Grid positions are labelled by i,. j and k indices, e.g. $p(i,j,k)$. The state vector has n elements in total.

A weather balloon makes a single observation of temperature which exactly coincides with the grid point at $(\tilde{i},\tilde{j},\tilde{k})$. It has value y and error variance σ_{O}^2.

(a) What is an innovation vector? Write down the innovation for this system. [2 marks]

(b) Let $\mathbf{H}$ be the Jacobian of the observation operator in this case. Define a structure for $\vec{x}$ and let its rth element correspond to the temperature at position $(\tilde{i},\tilde{j},\tilde{k})$. Write down $\mathbf{H}$ and $\mathbf{H}^{\mathrm{T}}$ stating the size of each matrix. [5 marks]

(c) Let the $\mathbf{B}$-matrix have the following form

$$\mathbf{B} = \begin{pmatrix} B_{11} & \cdots & B_{1r} & \cdots & B_{1n} \\ \vdots & \ddots & \vdots & ⋰ & \vdots \\ B_{r1} & \cdots & B_{rr} & \cdots & B_{rn} \\ \vdots & ⋰ & \vdots & \ddots & \vdots \\ B_{n1} & \cdots & B_{nr} & \cdots & B_{nn} \end{pmatrix},$$

Write down the result of the matrix operations

i) $\mathbf{BH}^{\mathrm{T}}$ and [2 marks]

ii) $\mathbf{HBH}^{\mathrm{T}}$. [2 marks]

(d) Use the equivalence between 3d-Var. and optimal interpolation in this case and write the analysis increment that follows from assimilating the temperature observation. [4 marks]

(e) An assimilation expert wishes to probe the structure of the $\mathbf{B}$-matrix. He has no previous knowledge of the $\mathbf{B}$-matrix that is used in the 3d-Var. system, and treats the assimilation system as a black box. With the aid of your answer from (d), explain how information from the above assimilation run can help the expert understand part of the structure of the $\mathbf{B}$-matrix.

[3 marks]

(f) With reference to your answer from (e), how would it possible for the single temperature observation to affect the analysed state at grid points away from $(\tilde{i},\tilde{j},\tilde{k})$ and of the other variables q and p? [3 marks]

(g) The above assimilation runs do not make full use of all the advantages of 3d-Var. over non-variational methods of data assimilation. List some advantages of the variational method over optimal interpolation. What kind of observational data would demonstrate the advantages more fully? [3 marks]

(h) Assuming that all error statistics are specified correctly, that the background and observation have no biases, that the observation operator is accurate and that the assimilation has converged, what value would you expect J to have at the analysis in the 3d-Var. run? [1 mark]