The Implementation of Potential Vorticity as a Leading Control Variable in Var.

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Why not use model variables as control parameters?

\[
J[\tilde{x}] = J_B + J_O \\
= \frac{1}{2} \tilde{x}'^T B^{-1} \tilde{x}' + \frac{1}{2} (\tilde{H}[\tilde{x}' + \tilde{x}_B] - \tilde{y})^T R^{-1} (\tilde{H}[\tilde{x} + \tilde{x}_B] - \tilde{y})
\]

where \( \tilde{x}' = \tilde{x} - \tilde{x}_B \)

- \( B \) (in \( \tilde{x} \)-space) contains \( > 10^{14} \) elements and cannot be represented explicitly.
- \( B \) (in \( \tilde{x} \)-space) is badly conditioned
  \[ \text{max e.v. / min e.v.} \approx 10^{10}. \]

BADLY CONDITIONED (model space)
Solution: for variational data assimilation vary weights of the eigenvectors of $B$ (instead of components of $\tilde{x}$).

$$\tilde{\nu} = U^{-1} \tilde{x}$$
$$U^{-1} = \Lambda^{-1/2} L^T$$
$$B = UU^T$$

$\Lambda$ diagonal matrix of e.values, columns of $L$ are e.functions.

$$J[\tilde{\nu}] = \frac{1}{2} \tilde{\nu}^T \tilde{\nu} + \frac{1}{2} (\tilde{H}[U\tilde{\nu} + \tilde{x}_b] - \tilde{y})^T R^{-1} (\tilde{H}[U\tilde{\nu} + \tilde{x}_b] - \tilde{y})$$

This problem is much better conditioned.

But, this can't be done directly

$$\tilde{\nu} = U^{-1} \tilde{x}$$
$$= U_h^{-1} U_v^{-1} U_p^{-1} \tilde{x}$$

$\uparrow$ ★ parameter transform ★

$\uparrow$ vertical transform

$\uparrow$ horizontal transform

The role of $U_p^{-1}$ is to 'block-diagonalize' the multi-variate correlations.

What parameters?
Existing scheme (pragmatic/engineering approach)

<table>
<thead>
<tr>
<th>Subspace Parameter</th>
<th>( \psi )</th>
<th>Captures most of the flow …</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \chi )</td>
<td></td>
<td>Captures most of the rest of the flow …</td>
</tr>
<tr>
<td>( \lambda_p )</td>
<td></td>
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These are orthogonal but not uncorrelated

Proposed scheme (physics approach)

<table>
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<th>Subspace Parameter</th>
<th>( s )</th>
<th>&quot;Balanced&quot; / &quot;slow manifold&quot; (PV) …</th>
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<td>( \psi )</td>
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- These are orthogonal, but are expected to be only weakly correlated.
- The new parameters are thought to evolve independently, each occupying a separate region in normal mode space.

Why not assimilate using only the leading parameter?
What grid staggering for new parameters?

\[ PV, \quad \bar{PV}, \quad \nabla \cdot \vec{v}, \quad s, \quad \frac{U}{p}, \quad \chi \]

Met Office Var. Grid Staggering
(black: model variables; red: existing Var. parameters)

Charney-Phillips in vertical
Arakawa C in horizontal

There is one more full level than half levels