Diagonal preconditioning of the PV equations for preconditioning

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The following equations are based on Bannister/Cullen, but with the following simplifications:

- Discrete effects are ignored (continuous equations used).
- Derivatives of some terms are ignored.

1. The balanced transforms

1.1 The balanced U-transform

Potential vorticity in the GCR solver is calculated via streamfunction and pressure on ψ-points. Effectively, the balanced **U**-transform can be written as the following which has been simplified by ignoring the spatial derivatives of $f\rho_0$,

$$\vec{X}_1' = \begin{pmatrix} 1 \\ f\rho_0 \end{pmatrix} \psi_b', \tag{1}$$

(this is (22) of [1]). Balanced streamfunction (first element) translates to (balanced) vorticity,

$$\begin{aligned} \zeta' &= \nabla^2 \psi_b', \\ &= \frac{1}{r^2 \cos\phi} \left(\frac{1}{\cos\phi} \frac{\partial^2 \psi_b'}{\partial \lambda^2} + \frac{\partial}{\partial \phi} \left(\cos\phi \frac{\partial \psi_b'}{\partial \phi} \right) \right). \end{aligned} \tag{2}$$

1.2 Increment conversions and the PV formula

In order to derive a simpler formula for the PV, we need some relations between variables. In particular, we need formulae for ρ' and θ_z' in terms of pressure increments (and vertical derivatives of). Start with some equations from Appendix A of [1] (before discretization is considered),

$$\rho' = \frac{1-\kappa}{R\Pi_0\theta_0} p' - \frac{\rho_0}{\theta_0}\theta', \qquad \times \frac{\partial \Pi_0}{\partial z} \tag{3}$$

$$\theta_0 \frac{\partial}{\partial z}\left(\kappa \frac{\Pi_0}{p_0} p' \right) + \frac{\partial \Pi_0}{\partial z}\theta' = 0, \qquad \times \frac{\rho_0}{\theta_0} \tag{4}$$

where (3) is (47) of [1] and (4) is (48) of [1]. Making the multiplications shown, eliminating θ', and not paying attention to the discretization gives an expression for ρ',

$$(3) \Rightarrow \frac{\partial \Pi_0}{\partial z}\rho' = \frac{\partial \Pi_0}{\partial z} \frac{1-\kappa}{R\Pi_0\theta_0} p' - \frac{\partial \Pi_0}{\partial z}\frac{\rho_0}{\theta_0}\theta',$$

$$(4) \Rightarrow \rho_0 \frac{\partial}{\partial z}\left(\kappa \frac{\Pi_0}{p_0} p' \right) + \frac{\rho_0}{\theta_0}\frac{\partial \Pi_0}{\partial z}\theta' = 0,$$

$$(add) \Rightarrow \frac{\partial \Pi_0}{\partial z}\rho' = \frac{\partial \Pi_0}{\partial z} \frac{1-\kappa}{R\Pi_0\theta_0} p' + \rho_0 \frac{\partial}{\partial z}\left(\kappa \frac{\Pi_0}{p_0} p' \right),$$

$$\begin{aligned} \rho' &= \frac{1-\kappa}{R\Pi_0\theta_0} p' + \left(\frac{\partial \Pi_0}{\partial z} \right)^{-1} \rho_0 \frac{\partial}{\partial z}\left(\kappa \frac{\Pi_0}{p_0} p' \right), \\ &= \frac{1-\kappa}{R\Pi_0\theta_0} p' + \left(\frac{\partial \Pi_0}{\partial z} \right)^{-1} \rho_0 \kappa \left(\frac{1}{p_0}\frac{\partial \Pi_0}{\partial z} p' - \frac{\Pi_0}{p_0^2}\frac{\partial p_0}{\partial z} p' + \frac{\Pi_0}{p_0} p_z' \right), \end{aligned}$$

$$= \frac{1-\kappa}{R\Pi_0\theta_0}p' + \rho_0\kappa\frac{1}{p_0}p' - \left(\frac{\partial\Pi_0}{\partial z}\right)^{-1}\rho_0\kappa\frac{\Pi_0}{p_0^2}\frac{\partial p_0}{\partial z}p' + \left(\frac{\partial\Pi_0}{\partial z}\right)^{-1}\rho_0\kappa\frac{\Pi_0}{p_0}p'_z,$$

$$= \left(\frac{1-\kappa}{R\Pi_0\theta_0} + \frac{\rho_0\kappa}{p_0}\right)p' + \frac{\Pi_0\rho_0\kappa}{p_0}\left(\frac{\partial\Pi_0}{\partial z}\right)^{-1}\left(-\frac{1}{p_0}\frac{\partial p_0}{\partial z}p' + p'_z\right). \qquad (5)$$

Next, $R\Pi_0\theta_0 = p_0/\rho_0$,

$$\rho' = \frac{\rho_0}{p_0}(1-\kappa+\kappa)p' + \frac{\Pi_0\rho_0\kappa}{p_0}\left(\frac{\partial\Pi_0}{\partial z}\right)^{-1}\left(-\frac{1}{p_0}\frac{\partial p_0}{\partial z}p' + p'_z\right). \qquad (6)$$

Also,

$$\frac{\partial\Pi_0}{\partial z} = \frac{\Pi_0\kappa}{p_0}\frac{\partial p_0}{\partial z}, \qquad (7)$$

(this is (40) of [1]). Eliminating $\partial\Pi_0/\partial z$ between (7) and the middle term only of the previous equation gives,

$$\rho' = \frac{\rho_0}{p_0}p' - \frac{1}{p_0}\frac{\Pi_0\rho_0\kappa}{p_0}\left(\frac{\partial\Pi_0}{\partial z}\right)^{-1}\frac{\partial p_0}{\partial z}p' + \frac{\Pi_0\rho_0\kappa}{p_0}\left(\frac{\partial\Pi_0}{\partial z}\right)^{-1}p'_z,$$

$$= \frac{\rho_0}{p_0}p' - \frac{\rho_0}{p_0}p' + \frac{\Pi_0\rho_0\kappa}{p_0}\left(\frac{\partial\Pi_0}{\partial z}\right)^{-1}p'_z,$$

$$\rho' = \frac{\Pi_0\rho_0\kappa}{p_0}\left(\frac{\partial\Pi_0}{\partial z}\right)^{-1}p'_z, \qquad (8)$$

where we are not worrying about vertical or horizontal interpolation.

The incremental form of $\partial\theta'/\partial z$ has the form,

$$\frac{\partial\theta'}{\partial z} = \frac{g}{c_p}\left\{\left(\frac{\partial\Pi_0}{\partial z}\right)^{-2}\frac{\partial^2}{\partial z^2}\left(\frac{\kappa\Pi_0}{p_0}p'\right) - 2\left(\frac{\partial\Pi_0}{\partial z}\right)^{-3}\frac{\partial^2\Pi_0}{\partial z^2}\frac{\partial}{\partial z}\left(\frac{\kappa\Pi_0}{p_0}p'\right)\right\}. \qquad (9)$$

1.3 The combined transform

All formulae have now been written in terms of increments of vorticity and pressure (or first and second vertical derivatives of pressure). The next stage is to substitute the three components from the balanced U-transform into the three terms for PV. The vorticity can be written directly in terms of the streamfunction, ψ'_b. The PV formula becomes (see (15) of [1]),

$$PV' = \frac{\theta_{0z}}{\rho_0}\zeta' - \frac{f\theta_{0z}}{\rho_0}\frac{\Pi_0\kappa}{p_0}\bar{\Pi}_{0z}^{-1}p'_z + \frac{f}{\rho_0}\frac{g}{c_p}\left\{\bar{\Pi}_{0z}^{-2}\frac{\partial^2}{\partial z^2}\left(\frac{\kappa\Pi_0}{p_0}p'\right) - 2\bar{\Pi}_{0z}^{-3}\Pi_{0zz}\frac{\partial}{\partial z}\left(\frac{\kappa\Pi_0}{p_0}p'\right)\right\}, \qquad (10)$$

$$= \frac{\theta_{0z}}{\rho_0}\frac{1}{r^2\cos\phi}\left(\frac{1}{\cos\phi}\frac{\partial^2\psi'_b}{\partial\lambda^2} + \frac{\partial}{\partial\phi}\left(\cos\phi\frac{\partial\psi'_b}{\partial\phi}\right)\right) -$$

$$\frac{f^2\theta_{0z}}{\rho_0}\frac{\Pi_0\kappa}{p_0}\bar{\Pi}_{0z}^{-1}\frac{\partial(\rho_0\psi'_b)}{\partial z} +$$

$$\frac{f^2}{\rho_0}\frac{g}{c_p}\left\{\bar{\Pi}_{0z}^{-2}\frac{\partial^2}{\partial z^2}\left(\frac{\kappa\Pi_0\rho_0}{p_0}\psi'_b\right) - 2\bar{\Pi}_{0z}^{-3}\Pi_{0zz}\frac{\partial}{\partial z}\left(\frac{\kappa\Pi_0\rho_0}{p_0}\psi'_b\right)\right\}. \qquad (11)$$

1.4 Discretized form of the equations (extract diagonal terms only)

The equations are discretized in the following way (we are trying to reflect these - approximately - in our extraction of diagonal points):

- The vorticity terms are computed on ψ-points directly. Factors (arising from zonal mean reference state quantities, e.g., are not interpolated in the horizontal).

In order to discretize and extract diagonal elements, note the following formulae,

(i) Discrete second derivatives,

$$\frac{\partial^2\psi(i,j,k)}{\partial x^2} = \frac{\psi(i+1,j,k)+\psi(i-1,j,k)-2\psi(i,j,k)}{\delta_x^2},$$

$$\frac{\partial}{\partial y}\left(\cos\phi\frac{\partial\psi}{\partial y}\right) = \frac{1}{\delta_y^2}\{\cos\phi_u(j+1)(\psi(i,j+1,k)-\psi(i,j,k))$$

$$-\cos\phi_u(j)(\psi(i,j,k)-\psi(i,j-1,k))\}.$$

(ii) Vertical interpolation from θ to p-points,

$$p(k) = \alpha_1(k)p_\theta(k)+\beta_1(k)p_\theta(i,j,k-1).$$

(iii) Vertical derivative of ψ-point quantity, followed by vertical interpolation from θ to p-levels,

$$\frac{\partial(\rho_0\psi_b')}{\partial z} = \alpha_1(k)\frac{\rho_0(k+1)\psi_b'(i,j,k+1)-\rho_0(k)\psi_b'(i,j,k)}{r_p(k+1)-r_p(k)}+$$

$$\beta_1(k)\frac{\rho_0(k)\psi_b'(i,j,k)-\rho_0(k-1)\psi_b'(i,j,k-1)}{r_p(k)-r_p(k-1)}.$$

If computing PV at the top and bottom (grid-only PV), then Neumann boundary conditions apply. At the top only the β-term exists, and at the bottom, only the α-term exists.

(iv) Second vertical derivative of ψ-point quantity,

$$\frac{\partial^2(\rho_0\psi_b')}{\partial z^2} = \frac{1}{r_\theta(k)-r_\theta(k-1)}\times$$

$$\left(\frac{\rho_0(k+1)s'(i,j,k+1)-\rho_0(k)\psi_b'(i,j,k)}{r_p(k+1)-r_p(k)}-\frac{\rho_0(k)s'(i,j,k)-\rho_0(k-1)\psi_b'(i,j,k-1)}{r_p(k)-r_p(k-1)}\right).$$

If computing PV at the top and bottom (grid-only PV), then Neumann boundary conditions apply. At the top only the second term exists, and at the bottom, only the first term exists.

The diagonal form of the combined bulk PV transform (on a ψ-point) is then,

$$PV'(i,j,k) \sim \left[\frac{\theta_{0z}(j,k)}{\rho_0(j,k)}\frac{1}{r^2\cos\phi_v(j)}\left(\frac{-2}{\delta_\lambda^2\cos\phi_v(j)}-\frac{\cos\phi_u(j+1)+\cos\phi_u(j)}{\delta_\phi^2}\right)-\right.$$

$$\frac{f^2\theta_{0z}(j,k)\Pi_0(j,k)\kappa\bar{\Pi}_{0z}^{-1}(j,k)}{p_0(j,k)}\left(\frac{-\alpha_1(i,j,k)}{r_p(i,j,k+1)-r_p(i,j,k)}+\frac{\beta_1(i,j,k)}{r_p(i,j,k)-r_p(i,j,k-1)}\right)+$$

$$\frac{f^2g\bar{\Pi}_{0z}^{-2}(j,k)}{c_p}\frac{\kappa\Pi_0(j,k)/p_0(j,k)}{r_\theta(i,j,k)-r_\theta(i,j,k-1)}\left(\frac{-1}{r_p(i,j,k+1)-r_p(i,j,k)}-\frac{1}{r_p(i,j,k)-r_p(i,j,k-1)}\right)-$$

$$\left.\frac{2gf^2\bar{\Pi}_{0z}^{-3}(j,k)\Pi_{0zz}(j,k)}{c_p}\frac{\kappa\Pi_0(j,k)}{p_0(i,j,k)}\left(\frac{-\alpha_1(i,j,k)}{r_p(i,j,k+1)-r_p(i,j,k)}+\frac{\beta_1(i,j,k)}{r_p(i,j,k)-r_p(i,j,k-1)}\right)\right]\psi_b'(i,j,k).$$

Do not forget to consider the effect of the Neumann boundary conditions at the top and bottom of the domain - see subsections (iii) and (iv) above.

2. The unbalanced transforms

Given an unbalanced parameter, ${}^u p$, the vorticity can be computed by setting $PV = 0$. These problems are treated in turn below (the bulk problem must be performed first as results are used for the top and bottom levels).

2.1 The unbalanced U-transform

Setting bulk PV to zero (using the PV formula from section 1.3 of this report) yields the following for the unbalanced vorticity,

$${}^u\zeta_p' = f\frac{\Pi_0\kappa}{p_0}\Pi_{0z}^{-1}\,{}^u p_z' - \frac{f}{\theta_{0z}}\frac{g}{c_p}\left\{\Pi_{0z}^{-2}\frac{\partial^2}{\partial z^2}\left(\frac{\kappa\Pi_0}{p_0}\,{}^u p'\right)-2\Pi_{0z}^{-3}\Pi_{0zz}\frac{\partial}{\partial z}\left(\frac{\kappa\Pi_0}{p_0}\,{}^u p'\right)\right\}, \tag{12}$$

where ${}^u\zeta'$ is the unbalanced vorticity increment and ${}^u p'$ is the unbalanced pressure (control variable). ${}^u\zeta'$ is to computed on all levels except the top and bottom, and ${}^{u}p'$ is known on all levels.

2.2 The anti-PV formula

The $\overline{PV}$ formula is Eq. (30), ignoring spatial derivatives of $f\rho_0$,

$$\overline{PV} = f\rho_0\,{}^{u}\zeta' - \nabla^2\,{}^{u}p'. \tag{13}$$

2.3 Discretization of the bulk equation (extract diagonal terms only)

The appropriate discretisation formula should be used from section 1.4, and the following.

(i) Vertical derivative of a p-point quantity, followed by vertical interpolation from a θ-level to a p-level, multiplied by a p-point quantity,

$$\rho_0\overline{\frac{\partial p'}{\partial z}} = \rho_0(i,j,k)p_{zp}(i,j,k),$$

$$= \rho_0(i,j,k)[\alpha_1(k)p_z(i,j,k) + \beta_1(k)p_z(i,j,k-1)],$$

$$= \rho_0(i,j,k)[\alpha_1(k)\frac{p(i,j,k+1)-p(i,j,k)}{r_p(k+1)-r_p(k)} + \beta_1(k)\frac{p(i,j,k)-p(i,j,k-1)}{r_p(k)-r_p(k-1)}].$$

If computing unbalanced vorticity at the top and bottom (grid-only PV), then Neumann boundary conditions apply. At the top only the β-term exists, and at the bottom, only the α-term exists.

(ii) Second vertical derivative of a p-point quantity, multiplied by a p-point quantity,

$$\rho_0\frac{\partial^2 p'}{\partial z^2} = \rho_0(i,j,k)p_{zz}(i,j,k),$$

$$= \rho_0(i,j,k)\frac{1}{r_\theta(k)-r_\theta(k-1)}\left[\frac{p(i,j,k+1)-p(i,j,k)}{r_p(k+1)-r_p(k)} - \frac{p(i,j,k)-p(i,j,k-1)}{r_p(k)-r_p(k-1)}\right].$$

If computing unbalanced vorticity at the top and bottom (grid-only PV), then Neumann boundary conditions apply. At the top only the second term exists, and at the bottom, only the first term exists.

(iii) Horizontal Laplacian of a quantity on p-points.

$$\nabla^2 p(i,j) = \frac{p(i+1,j)-2p(i,j)+p(i-1,j)}{\delta_\lambda^2 r^2\cos^2\phi_u(j)} + \frac{[p(i,j+1)-p(i,j)]\cos\phi_v(j) - [p(i,j)-p(i,j-1)]\cos\phi_v(j-1)}{\delta_\phi^2 r^2\cos\phi_u(j)}.$$

The zero PV transform, taking diagonal terms only becomes,

$${}^{u}\zeta'_p(i,j,k) = f\kappa\frac{\Pi_0(i,j,k)}{p_0(i,j,k)\Pi_{0z}(i,j,k)}\left\{\frac{\beta_1(k)}{r_p(k)-r_p(k-1)} - \frac{\alpha_1(k)}{r_p(k+1)-r_p(k)}\right\}{}^{u}p'(i,j,k) +$$

$$\frac{fg\kappa}{c_p}\frac{1}{\theta_{0z}(i,j,k)\Pi_{0z}^2(i,j,k)}\frac{1}{r_\theta(k)-r_\theta(k-1)}\left[\frac{1}{r_p(k+1)-r_p(k)} + \frac{1}{r_p(k)-r_p(k-1)}\right]\frac{\Pi_0(i,j,k)\,{}^{u}p'(i,j,k)}{p_0(i,j,k)} +$$

$$\frac{2fg\kappa}{c_p}\frac{\Pi_{0zz}(i,j,k)}{\theta_{0z}(i,j,k)\Pi_{0z}^3(i,j,k)}\left\{\frac{\beta_1(k)}{r_p(k)-r_p(k-1)} - \frac{\alpha_1(k)}{r_p(k+1)-r_p(k)}\right\}\frac{\Pi_0(i,j,k)\,{}^{u}p'(i,j,k)}{p_0(i,j,k)}.$$

Do not forget to consider the effect of the Neumann boundary conditions at the top and bottom of the domain - see subsections (i) and (ii) above. It is sufficient to take the diagonal part at this stage (ie before substituting into the $\overline{PV}$), because the $\overline{PV}$ formula does not differentiate the unbalanced vorticity.

Substituting this into the $\overline{PV}$ gives, along with the pressure Laplacian,

$$\overline{PV}(i,j,k) = f\rho_0(i,j,k)\times$$

$$\left[\left\{\frac{\beta_1(k)}{r_p(k)-r_p(k-1)}-\frac{\alpha_1(k)}{r_p(k+1)-r_p(k)}\right\}\left\{f\kappa\frac{\Pi_0(i,j,k)}{p_0(i,j,k)\Pi_{0z}(i,j,k)}+\frac{2fg\kappa}{c_p}\frac{\Pi_{0zz}(i,j,k)}{\theta_{0z}(i,j,k)\Pi_{0z}^3(i,j,k)}\frac{\Pi_0(i,j,k)}{p_0(i,j,k)}\right\}+\right.$$

$$\left.\frac{fg\kappa}{c_p}\frac{1}{\theta_{0z}(i,j,k)\Pi_{0z}^2(i,j,k)}\frac{1}{r_\theta(k)-r_\theta(k-1)}\left[\frac{1}{r_p(k+1)-r_p(k)}+\frac{1}{r_p(k)-r_p(k-1)}\right]\frac{\Pi_0(i,j,k)}{p_0(i,j,k)}\right]{}^u p'(i,j,k)-$$

$$\frac{1}{r^2}\left\{-\frac{2}{\delta_\lambda^2\cos^2\phi_u(j)}-\frac{\cos\phi_v(j)+\cos\phi_v(j-1)}{\delta_\phi^2\cos\phi_u(j)}\right\}{}^u p'(i,j,k).$$

Do not forget to consider the effect of the Neumann boundary conditions at the top and bottom of the domain - see subsections (i) and (ii) above.

Reference

[1] Bannister R.N., Cullen M.J.P., 2008, New PV-based variables for Met Office VAR, Version 07.