Particle Filter Formulae
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Bayes theorem
('prior' to 'posterior')

\[
p(\psi^n \mid d^n) = \frac{1}{A^n} p(d^n \mid \psi^n) p(\psi^n), \tag{1}
\]

\text{(time level } n\text{).}

'Prior' in particle form

\[
p(\psi^n) = \frac{1}{N} \sum_{i=1}^{N} w^n_i \delta (\psi^n - \psi^n_i), \tag{2}
\]

\[
\sum_{i=1}^{N} w^n_i = N.
\]

'Posterior' in particle form

\[
p(\psi^n \mid d^n) = \frac{1}{N} \sum_{i=1}^{N} \tilde{w}^n_i \delta (\psi^n - \psi^n_i), \tag{3}
\]

\[
\sum_{i=1}^{N} \tilde{w}^n_i = N.
\]

Bayes theorem in particle form

Substitute (2) and (3) in (1)

\[
\frac{1}{N} \sum_{i=1}^{N} \tilde{w}^n_i \delta (\psi^n - \psi^n_i) = \frac{1}{A^n} p(d^n \mid \psi^n) \frac{1}{N} \sum_{i=1}^{N} w^n_i \delta (\psi^n - \psi^n_i),
\]

Multiply by \(\delta (\psi^n - \psi^n_i)\) and integrate over all \(\psi^n\)

\[
\frac{1}{N} \sum_{i=1}^{N} \tilde{w}^n_i \int d\psi^n \delta (\psi^n - \psi^n_i) \delta (\psi^n - \psi^n_i) =
\]

\[
\frac{1}{A^n} \int d\psi^n p(d^n \mid \psi^n) \frac{1}{N} \sum_{i=1}^{N} w^n_i \delta (\psi^n - \psi^n_i) \delta (\psi^n - \psi^n_i),
\]

\[
\therefore \tilde{w}^n_i = \frac{1}{A^n} p(d^n \mid \psi^n_i) w^n_i. \tag{4}
\]

Particles don’t change their positions in this step - just their weights.

Transition density ('forecast' of 'prior')

\[
p(\psi^n) = \int d\psi^{n-1} p(\psi^n \mid \psi^{n-1}) p(\psi^{n-1}), \tag{5}
\]

where \(p(\psi^n \mid \psi^{n-1})\) is the transition density. For a deterministic model

\[
\psi^n = f (\psi^{n-1}),
\]

the transition density is a delta-function

\[
p(\psi^n \mid \psi^{n-1}) = \delta (\psi^n - f (\psi^{n-1})),
\]

\[
\therefore p(\psi^n) = \int d\psi^{n-1} \delta (\psi^n - f (\psi^{n-1})) p(\psi^{n-1}). \tag{5a}
\]

For a stochastic model
\[ \psi^n = f(\psi^{n-1}) + \beta^n, \]

where \( \beta^n \) is a random variable with distribution \( r(\beta^n) \), the transition density is

\[ p(\psi^n | \psi^{n-1}) = r(\psi^n - f(\psi^{n-1})), \]

\[ \therefore p(\psi^n) = \int d\psi^{n-1} r(\psi^n - f(\psi^{n-1})) p(\psi^{n-1}). \tag{5b} \]

**Proposal density**

The transition density may be written as follows

\[ p(\psi^n | \psi^{n-1}) = q(\psi^n | \psi^{n-1}, d^n) \frac{p(\psi^n | \psi^{n-1})}{q(\psi^n | \psi^{n-1}, d^n)}, \tag{6} \]

where \( q(\psi^n | \psi^{n-1}, d^n) \) is a chosen function (called the proposal density) that has support greater than or equal to that of \( p(\psi^n | \psi^{n-1}) \). It is possible to replace the transition density with the proposal density

\[ p(\psi^n | \psi^{n-1}) \rightarrow q(\psi^n | \psi^{n-1}, d^n), \tag{7} \]

as long as the compensation term, \( p(\psi^n | \psi^{n-1}) / q(\psi^n | \psi^{n-1}, d^n) \), multiplies \( p(\psi^{n-1}) \)

\[ p(\psi^{n-1}) \rightarrow \frac{p(\psi^n | \psi^{n-1})}{q(\psi^n | \psi^{n-1}, d^n)} p(\psi^{n-1}). \tag{8} \]

This can be seen by substituting (6) into (5) to give

\[ p(\psi^n) = \int d\psi^{n-1} q(\psi^n | \psi^{n-1}, d^n) \left( \frac{p(\psi^n | \psi^{n-1})}{q(\psi^n | \psi^{n-1}, d^n)} \right) p(\psi^{n-1}) \right). \tag{9} \]

\( q(\psi^n | \psi^{n-1}, d^n) \) may be chosen to depend upon the observations if required.

The equivalent picture in terms of particle weights emerges by substituting (2) into (8)

\[ \sum_{i=1}^{N} w_i^{n-1} \delta(\psi_i^{n-1} - \psi_i^{n-1}) \rightarrow \frac{p(\psi^n | \psi^{n-1})}{q(\psi^n | \psi^{n-1}, d^n)} \sum_{i=1}^{N} w_i^{n-1} \delta(\psi_i^{n-1} - \psi_i^{n-1}), \]

multiplying by \( \delta(\psi^n - \psi^{n-1}) \) and integrating over all \( \psi^{n-1} \)

\[ \sum_{i=1}^{N} w_i^{n-1} \int d\psi^{n-1} \delta(\psi_i^{n-1} - \psi_i^{n-1}) \delta(\psi_i^{n-1} - \psi_i^{n-1}) \rightarrow \]

\[ \int d\psi^{n-1} \frac{p(\psi^n | \psi^{n-1})}{q(\psi^n | \psi^{n-1}, d^n)} \sum_{i=1}^{N} w_i^{n-1} \delta(\psi_i^{n-1} - \psi_i^{n-1}) \delta(\psi_i^{n-1} - \psi_i^{n-1}), \]

\[ \therefore w_i^{n-1} \rightarrow \frac{p(\psi^n | \psi_i^{n-1})}{q(\psi^n | \psi_i^{n-1}, d^n)} w_i^{n-1}. \tag{10} \]