PDF symmetry of difference between two iid variables

Ross Bannister

May 29, 2013

Suppose that two variables $x_A$ and $x_B$ are drawn from independent and identical (but otherwise arbitrary) distributions. Let this common distribution be $p(x)$:

$$
\begin{align*}
  x_A & \text{ drawn from } p(x_A), \\
  x_B & \text{ drawn from } p(x_B).
\end{align*}
$$

Ingelby et al (2012) (Sec. 2.3) states that if two variables are drawn from identical distributions then their differences must be from a symmetric distribution. No condition of symmetry is imposed on the identical distributions, $p(x)$. A number of questions arises:

1. How do we prove that this is correct?

2. What is this symmetric distribution in terms of $p(x)$?

Let the difference be $y = x_B - x_A$ and let the probability of getting a particular difference, $y$, for a specific set of variables $x_A$ and $x_B$ be $p_{\text{diff}}(y, x_A, x_B)$:

$$
p_{\text{diff}}(y, x_A, x_B) = p(x_A)p(x_B)\delta(x_A - x_B - y).
$$

We want to know the probability of this difference, $y$, without the conditioning on $x_A$ and $x_B$. To calculate this (call this $p_{\text{diff}}(y)$), integrate over all values of $x_A$ and $x_B$:

$$
p_{\text{diff}}(y) = \int \int dx_A dx_B p_{\text{diff}}(y, x_A, x_B),
\quad
= \int \int dx_A dx_B p(x_A)p(x_B)\delta(x_A - x_B - y),
\quad
= \int \int dx p(x)p(x - y),
$$

where we have relabelled $x_B$ as $x$. Now let $v = x - y/2$ be an alternative variable for $x$. Then

$$
\begin{align*}
  dv &= dx, \\
  x &= v + y/2, \\
  x - y &= v - y/2,
\end{align*}
$$
\( p_{\text{diff}}(y) \) is then equivalent to

\[
p_{\text{diff}}(y) = \int \int dv \, p(v + y/2)p(v - y/2).
\]

The important property of this last result is that \( p_{\text{diff}}(y) = p_{\text{diff}}(-y) \), which is a symmetric distribution. It also says what the form of the symmetric distribution is.