The Radiative Transfer Equation

Ross Bannister, January/April 2007

Derivation of the radiative transfer equation

As a pencil of radiation traverses a layer of the atmosphere, the radiance is modified in three ways (acting to either increase (+) or decrease (-) the radiation).

- **Emission**. The air in the layer emits as a grey body according to its temperature and emission characteristics (+).
- **Absorption**. The air in the layer absorbs a fraction of the radiation traversing it (-).
- **Scattering**. The air in the layer scatters the a fraction of the radiation to another direction (-), and a fraction of other radiation is scattered into the pencil (+).

Here scattering processes will be ignored. Let the intensity of radiation at wavelength λ and at height s above the ground be I_λ (known as the monochromatic radiance). Radiance has dimensions of power/(wavelength $\times$ area $\times$ solid angle). The radiation enters the layer at height s and leaves the layer at height $s + \delta s$ (Fig. 1).

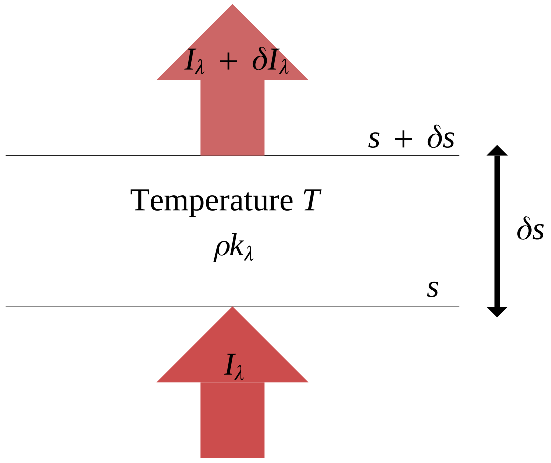

Figure 1: Radiation traversing a layer of the atmosphere.

Emission of radiation

The emission of radiance from the layer involves the Planck function, $B_\lambda(T)$, which is a function of the layer's temperature, T. B_λ has the same units as I_λ. Emission modifies the radiation according to

$$\delta I_\kappa^{emission} = \text{area of emitter} \times B_\lambda(T). \qquad (1)$$

Let k_λ be the air's absorption cross section (equivalent to the emission cross section by Kirchoff's law), which has units of area/mass. In a unit area of radiation into which the radiation passes, the mass of the layer is $\rho\delta s$ (where ρ is the air's density), and so the area of the emitter is then $k_\lambda \rho \delta s$. The contribution from emission is thus

$$\delta I_\kappa^{emission} = k_\lambda \rho \delta s\, B_\lambda(T). \qquad (2)$$

Absorption of radiation

The absorption of radiance from the layer is

$$\delta I_\kappa^{absoption} = -\,\text{area of absorber} \times I_\lambda, \qquad (3)$$

where the area of the absorber is the same as the area of the emitter in (1). Equation (3) then leads to

$$\delta I_{\kappa}^{absoption} = -k_{\lambda}\rho\delta s I_{\lambda}. \tag{4}$$

Equation (4) is sometimes known as Lambert's law.

The total change of radiation

The sum of (2) and (4) gives the combined effect, which gives a differential equation describing radiative transfer in the absence of scattering

$$\frac{\mathrm{d}\, I_{\lambda}}{\mathrm{d}\, s} = \rho k_{\lambda} \left(B_{\lambda}(T) - I_{\lambda}\right). \tag{5}$$

Integrating the radiative transfer equation

Multiplying (5) by the integrating factor $\exp \int_0^s \rho k_{\lambda}\ \mathrm{d}s'$, gives

$$\frac{\mathrm{d}\, I_{\lambda}}{\mathrm{d}\, s} \exp \int_0^s \rho k_{\lambda}\ \mathrm{d}s' = \rho k_{\lambda} \left(B_{\lambda}(T) \exp \int_0^s \rho k_{\lambda}\ \mathrm{d}s' - I_{\lambda} \exp \int_0^s \rho k_{\lambda}\ \mathrm{d}s'\right). \tag{6}$$

Note the following identity, found by the product rule for integration, which will be useful in rewriting (6)

$$\begin{aligned}
\frac{1}{\rho k_{\lambda}} \frac{\mathrm{d}}{\mathrm{d}\, s} \left(I_{\lambda} \exp \int_0^s \rho k_{\lambda}\ \mathrm{d}s'\right) &= \frac{1}{\rho k_{\lambda}} \left(\frac{\mathrm{d}\, I_{\lambda}}{\mathrm{d}\, s} \exp \int_0^s \rho k_{\lambda}\ \mathrm{d}s' + I_{\lambda} \frac{\mathrm{d}}{\mathrm{d}\, s} \exp \int_0^s \rho k_{\lambda}\ \mathrm{d}s'\right), \\
&= \frac{1}{\rho k_{\lambda}} \left(\frac{\mathrm{d}\, I_{\lambda}}{\mathrm{d}\, s} \exp \int_0^s \rho k_{\lambda}\ \mathrm{d}s' + I_{\lambda} \exp \int_0^s \rho k_{\lambda}\ \mathrm{d}s' \frac{\mathrm{d}}{\mathrm{d}\, s} \int_0^s \rho k_{\lambda}\ \mathrm{d}s'\right), \\
&= \frac{1}{\rho k_{\lambda}} \left(\frac{\mathrm{d}\, I_{\lambda}}{\mathrm{d}\, s} \exp \int_0^s \rho k_{\lambda}\ \mathrm{d}s' + I_{\lambda} \rho k_{\lambda} \exp \int_0^s \rho k_{\lambda}\ \mathrm{d}s'\right), \\
&= \left(\frac{1}{\rho k_{\lambda}} \frac{\mathrm{d}\, I_{\lambda}}{\mathrm{d}\, s} + I_{\lambda}\right) \exp \int_0^s \rho k_{\lambda}\ \mathrm{d}s'.
\end{aligned} \tag{7}$$

Equation (6) can therefore be written

$$\frac{\mathrm{d}}{\mathrm{d}\, s} \left(I_{\lambda} \exp \int_0^s \rho k_{\lambda}\ \mathrm{d}s'\right) = \rho k_{\lambda} B_{\lambda}(T) \exp \int_0^s \rho k_{\lambda}\ \mathrm{d}s'. \tag{8}$$

Integrating (8) from the bottom of the domain ($s = 0$) to height s

$$I_{\lambda}(s) \exp \int_0^s \rho k_{\lambda}\ \mathrm{d}s' - I_{\lambda}(0) = \int_0^s \rho k_{\lambda} B_{\lambda}(T(s'')) \left(\exp \int_0^{s''} \rho k_{\lambda}\ \mathrm{d}s'\right) \mathrm{d}s'', \tag{9}$$

where some reindexing has been performed for clarity. The term $I_{\lambda}(0)$ is the emission from the ground, which will depend upon the surface temperature. Taking the surface term to the right hand side and multiplying by $\exp - \int_0^s \rho k_{\lambda}\ \mathrm{d}s'$ gives an expression for the radiance at height s

$$I_{\lambda}(s) = I_{\lambda}(0) \exp - \int_0^s \rho k_{\lambda}\ \mathrm{d}s' + \int_0^s \rho k_{\lambda} B_{\lambda}(T(s'')) \left(\exp \int_0^{s''} \rho k_{\lambda}\ \mathrm{d}s' - \int_0^s \rho k_{\lambda}\ \mathrm{d}s'\right) \mathrm{d}s'', \tag{10}$$

$$= I_{\lambda}(0) \exp - \int_0^s \rho k_{\lambda}\ \mathrm{d}s' + \int_0^s \rho k_{\lambda} B_{\lambda}(T(s'')) \left(\exp - \int_{s''}^s \rho k_{\lambda}\ \mathrm{d}s'\right) \mathrm{d}s''. \tag{11}$$

An important radiance is the top-of-atmosphere radiance since this is measured by satellites

$$I_{\lambda}(\infty) = I_{\lambda}(0) \exp - \int_0^{\infty} \rho k_{\lambda}\ \mathrm{d}s' + \int_0^{\infty} \rho k_{\lambda} B_{\lambda}(T(s'')) \left(\exp - \int_{s''}^{\infty} \rho k_{\lambda}\ \mathrm{d}s'\right) \mathrm{d}s''. \tag{12}$$

The optical depth and the atmospheric transmittance

The optical depth, u, is a dimensionless unit of depth that determines the amount of attenuation that a pencil of radiation suffers. Optical depth between s_1 and s_2 is defined as follows

$$u(s_1,\ s_2) \ = \ \int_{s_1}^{s_2} \rho k_\lambda \ \mathrm{d}s'. \tag{13}$$

The attenuation between s_1 and s_2 is therefore

$$\exp -u(s_1,\ s_2)\,. \tag{14}$$

Radiation that traverses an optical depth of 1 will be reduced by a factor e. The atmospheric transmittance, $\tau(s)$, is defined as the attenuation factor that a pencil of radiation suffers as it traverses the atmosphere from height s to the top of the atmosphere

$$\tau(s) \ = \ \exp - \int_{s}^{\infty} \rho k_\lambda \ \mathrm{d}s', \tag{15}$$

$$= \ \exp -u(s,\ \infty)\,. \tag{16}$$

Another useful quantity is the derivative of τ with respect to s.

$$\begin{aligned}
\frac{\mathrm{d}\tau(s)}{\mathrm{d}s} &= \tau(s)\frac{\mathrm{d}}{\mathrm{d}s}\int_{\infty}^{s} \rho k_\lambda \ \mathrm{d}s', \\
&= \tau(s) \lim_{\delta s \to 0} \frac{1}{\delta s}\left(\int_{\infty}^{s+\delta s} \rho k_\lambda \ \mathrm{d}s' - \int_{\infty}^{s} \rho k_\lambda \ \mathrm{d}s'\right), \\
&= \tau(s) \lim_{\delta s \to 0} \frac{1}{\delta s}\int_{s}^{s+\delta s} \rho k_\lambda \ \mathrm{d}s', \\
&= \tau(s)\rho(s)k_\lambda(s)\,.
\end{aligned} \tag{17}$$

Using (16) and (17), the top-of-atmosphere radiance (12) can be written in a very compact form

$$\begin{aligned}
I_\lambda(\infty) &= I_\lambda(0)\tau(0) + \int_0^{\infty} \rho(s'')k_\lambda(s'')B_\lambda(T(s''))\tau(s'') \ \mathrm{d}s'', \\
&= I_\lambda(0)\tau(0) + \int_0^{\infty} B_\lambda(T(s''))\frac{\mathrm{d}\tau(s'')}{\mathrm{d}s''} \ \mathrm{d}s''.
\end{aligned} \tag{18}$$

The temperature weighting function

The weighting function with respect to the Planck function at height s is defined as $\mathrm{d}I_\lambda(\infty)/\ \mathrm{d}B(T(s))$. It can be found from (18)

$$\begin{aligned}
\frac{\mathrm{d}I_\lambda(\infty)}{\mathrm{d}B(T(s))} &= \frac{\mathrm{d}}{\mathrm{d}B(T(s))}\int_0^{\infty} B_\lambda(T(s''))\frac{\mathrm{d}\tau(s'')}{\mathrm{d}s''} \ \mathrm{d}s'', \\
&= \int_0^{\infty} \delta(s'' - s)\frac{\mathrm{d}\tau(s'')}{\mathrm{d}s''} \ \mathrm{d}s'', \\
&= \frac{\mathrm{d}\tau(s)}{\mathrm{d}s}.
\end{aligned} \tag{19}$$

The weighting function with respect to temperature at height s is defined as $\mathrm{d}I_\lambda(\infty)/\ \mathrm{d}T(s)$. It can be found from (19) and the chain rule

$$\frac{\mathrm{d}I_\lambda(\infty)}{\mathrm{d}T(s)} \ = \ \frac{\mathrm{d}I_\lambda(\infty)}{\mathrm{d}B(T(s))}\frac{\mathrm{d}B(T(s))}{\mathrm{d}T(s)} \ = \ \frac{\mathrm{d}\tau(s)}{\mathrm{d}s}\frac{\mathrm{d}B(T(s))}{\mathrm{d}T(s)}. \tag{20}$$