

# How to combine errors

Robin Hogan

June 2006

## 1 What is an “error”?

All measurements have uncertainties that need to be communicated along with the measurement itself. Suppose we make a measurement of temperature,  $\hat{T}$ , but the “true” temperature is  $T$ . In this case our *instantaneous* error is  $\varepsilon_T = \hat{T} - T$ . Obviously we don’t know the value of  $\varepsilon_T$  for any specific measurement (otherwise we could simply subtract it and report the true value) but we should be able to estimate the *root-mean-squared* error, given by

$$\Delta T = \sqrt{\overline{\varepsilon_T^2}}, \quad (1)$$

and then report our measurement in the form  $\hat{T} \pm \Delta T$  (for example  $T = 284.6 \pm 0.2$  K). In Eq. 1, the overbar denotes the mean taken over a large number of measurements by an identical instrument.

Usually the quantity  $\Delta T$  is referred to simply as the “error” in the measurement. This is a bit misleading and is easy to confuse with the instantaneous error; a better term would have been “uncertainty”, but “error” is in such common use that we had better stick with it. Just keep in mind that what we usually mean is the root-mean-squared error.

If the instantaneous errors have a Gaussian distribution (also known as a Normal or Bell-shaped distribution) then approximately 68% of the individual measurements will lie between  $T - \Delta T$  and  $T + \Delta T$ , and 95% of them between  $T - 2\Delta T$  and  $T + 2\Delta T$ . Be aware that sometimes errors are stated to indicate the “95% confidence interval”, in which case they are equal to  $2\Delta T$ . It should be noted that the “measurement” may well be the mean of a number of samples, in which case we might take the *standard error of the mean* as an estimate of the error  $\Delta T$ .

## 2 Errors for functions of one variable

Usually we will have a formula we want to use to derive a new variable from one or more of our measured variables. Section 3 describes the general case

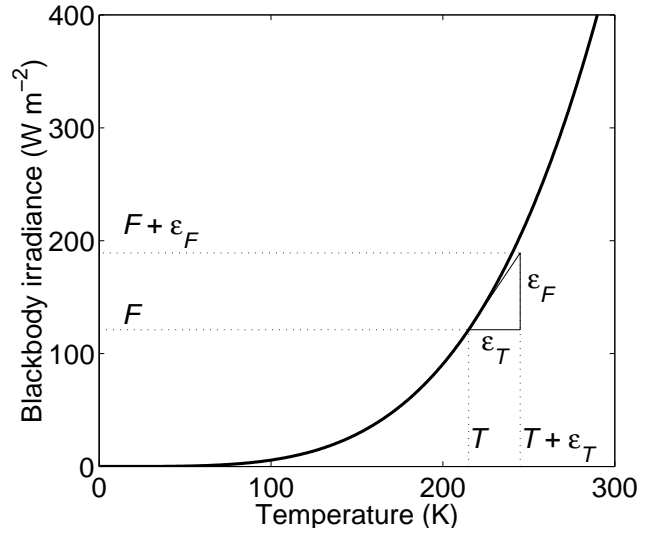


Figure 1: Illustration of the estimation of the error in  $F$  from the error in  $T$  using the gradient of the relationship between them (using Eq. 2).

in which we use the definition of an error in Eq. 1 to estimate the error in the new variable given the errors in the measured variables. However, if only one measurement is involved then we can use a simpler method based on differentiation.

Suppose we wish to derive the irradiance  $F$  emitted by a blackbody with a temperature  $T$  using

$$F = \sigma T^4, \quad (2)$$

where  $\sigma$  is a *constant* that is known very accurately. It can be seen from Fig. 1 that, provided the error in  $T$  is relatively small, the ratio of instantaneous errors in  $F$  and  $T$  is approximately equal to the gradient of the relationship between them:

$$\frac{\varepsilon_F}{\varepsilon_T} \simeq \frac{dF}{dT}. \quad (3)$$

From now on we will replace “ $\simeq$ ” with “ $=$ ”, but always be aware that error estimation is an approximate business so it is not worth quoting errors to high precision (certainly no more than two significant figures).

With the help of Eq. 1, it can be shown that the ratio of root-mean-squared errors is

$$\frac{\Delta F}{\Delta T} = \left| \frac{\varepsilon_F}{\varepsilon_T} \right| = \left| \frac{dF}{dT} \right|, \quad (4)$$

where  $|\cdot|$  denotes the absolute value (i.e. removing any minus sign) and is present because the root-mean-square of a real number is always positive. If we measure the temperature to be  $\hat{T} \pm \Delta T$  then we can use Eq. 4 to obtain the error in  $F$ :

$$\Delta F = \left| \frac{dF}{dT} \right| \Delta T = 4\sigma \hat{T}^3 \Delta T. \quad (5)$$

In the general case of a formula of the form  $a = f(b)$ , the error in  $a$  is given by

$$\Delta a = \left| \frac{df(b)}{db} \right| \Delta b. \quad (6)$$

The error formulae for some common functions have been calculated using Eq. 6 and are given below (where  $a$  and  $b$  are variables and  $\lambda$  and  $\mu$  are constants):

<i>Functions of one variable</i>	
$a = \lambda b$	$\Delta a =  \lambda  \Delta b$
$a = \lambda/b$	$\Delta a =  \lambda/b^2  \Delta b =  a/b  \Delta b$
$a = \lambda b^\mu$	$\Delta a =  \mu \lambda b^{\mu-1}  \Delta b =  \mu a/b  \Delta b$
$a = \lambda \exp(\mu b)$	$\Delta a =  \mu a  \Delta b$
$a = \lambda \ln(\mu b)$	$\Delta a =  \lambda/b  \Delta b$

(7)

### 3 Functions of two or more variables

In the general case, the quantity we want to calculate depends on more than one measurement, i.e.  $a = f(b, c, \dots)$ , and a more rigorous approach is needed to work out how the error in  $a$  depends on the errors in the other variables. Taking the simplest possible formula

$$a = b + c, \quad (8)$$

we replace  $a$  by  $\hat{a} + \varepsilon_a$ , etc., to obtain  $\hat{a} + \varepsilon_a = \hat{b} + \varepsilon_b + \hat{c} + \varepsilon_c$ . Noting that  $\hat{a} = \hat{b} + \hat{c}$ , the relationship between the instantaneous errors is then simply

$$\varepsilon_a = \varepsilon_b + \varepsilon_c. \quad (9)$$

From the definition of an error given in Eq. 1 we have

$$\begin{aligned} \Delta a &= \sqrt{\varepsilon_a^2} = \sqrt{(\varepsilon_b + \varepsilon_c)^2} \\ &= \sqrt{\varepsilon_b^2 + \varepsilon_c^2 + 2\varepsilon_b\varepsilon_c} \\ &= \sqrt{(\Delta b)^2 + (\Delta c)^2 + 2\varepsilon_b\varepsilon_c}. \end{aligned} \quad (10)$$

The term  $\overline{\varepsilon_b\varepsilon_c}$  is an *error covariance* and is zero provided that the measurements *independent*, i.e. their instantaneous errors are *uncorrelated*. Thus for independent measurements of  $b$  and  $c$ , the error formula is

$$\Delta a = \sqrt{(\Delta b)^2 + (\Delta c)^2}. \quad (11)$$

The error formulae for addition, subtraction, multiplication and division of independent variables can be derived in a similar manner:

<i>Functions of more than one variable</i>	
$a = b + c$ $a = b - c$	$(\Delta a)^2 = (\Delta b)^2 + (\Delta c)^2$
$a = bc$ $a = b/c$	$\left(\frac{\Delta a}{a}\right)^2 = \left(\frac{\Delta b}{b}\right)^2 + \left(\frac{\Delta c}{c}\right)^2$
$a = b + c + d$	$(\Delta a)^2 = (\Delta b)^2 + (\Delta c)^2 + (\Delta d)^2$
$a = bcd$	$(\Delta a/a)^2 = (\Delta b/b)^2 + (\Delta c/c)^2 + (\Delta d/d)^2$

(12)

### 4 Errors for more complicated formulae

For more complicated formulae we can combine the approaches in sections 2 and 3. For example, consider the formula

$$a = \lambda b^\mu \exp(c). \quad (13)$$

If we let  $a = xy$  where  $x = \lambda b^\mu$  and  $y = \exp(c)$ , then from Eq. 7 we know that  $\Delta x = |\mu x/b| \Delta b$  and  $\Delta y = |y| \Delta c$ . According to Eq. 12, we can combine these errors using the multiplication rule to obtain

$$\begin{aligned} \left(\frac{\Delta a}{a}\right)^2 &= \left(\frac{\Delta x}{x}\right)^2 + \left(\frac{\Delta y}{y}\right)^2 \\ &= \left(\frac{\mu \Delta b}{b}\right)^2 + (\Delta c)^2. \end{aligned} \quad (14)$$

and hence

$$\Delta a = a \left[ \left(\frac{\mu \Delta b}{b}\right)^2 + (\Delta c)^2 \right]^{\frac{1}{2}}. \quad (15)$$