A. Introduction to the Atmospheric Boundary Layer (ABL)

In 1904 the concept of a "boundary layer", where a moving fluid meets a solid boundary, was first described by *Ludwig Prandtl*. He showed that the flow could be split into two regions – the bulk of the flow where viscosity could be neglected (which was what most of the previous work had assumed), and the boundary layer around the surface of an object where viscosity was important, and where the tangential velocity of the flow to the surface fell rapidly to zero. This enabled the drag on objects to be understood, which was of particular use in the design of aircraft winds and the development of manned flight.

1. Definition and basic properties

A good working definition of the ABL:

- Usually around 1 km deep, but in mid-latitudes can vary from 100 m to 3 km.
- Temperatures vary diurnally, unlike the free atmosphere above.
- The surface influences the ABL by friction and by heat fluxes at the ground.
- Characterised by turbulence, which is generated by wind shear (wind is approximately geostrophic at the top of the ABL but zero at the surface). Temperature gradients can either generate or suppress turbulence.
- Boundary layer clouds: predominantly fair-weather cumulus, stratocumulus and fog.

2. Motivation for study

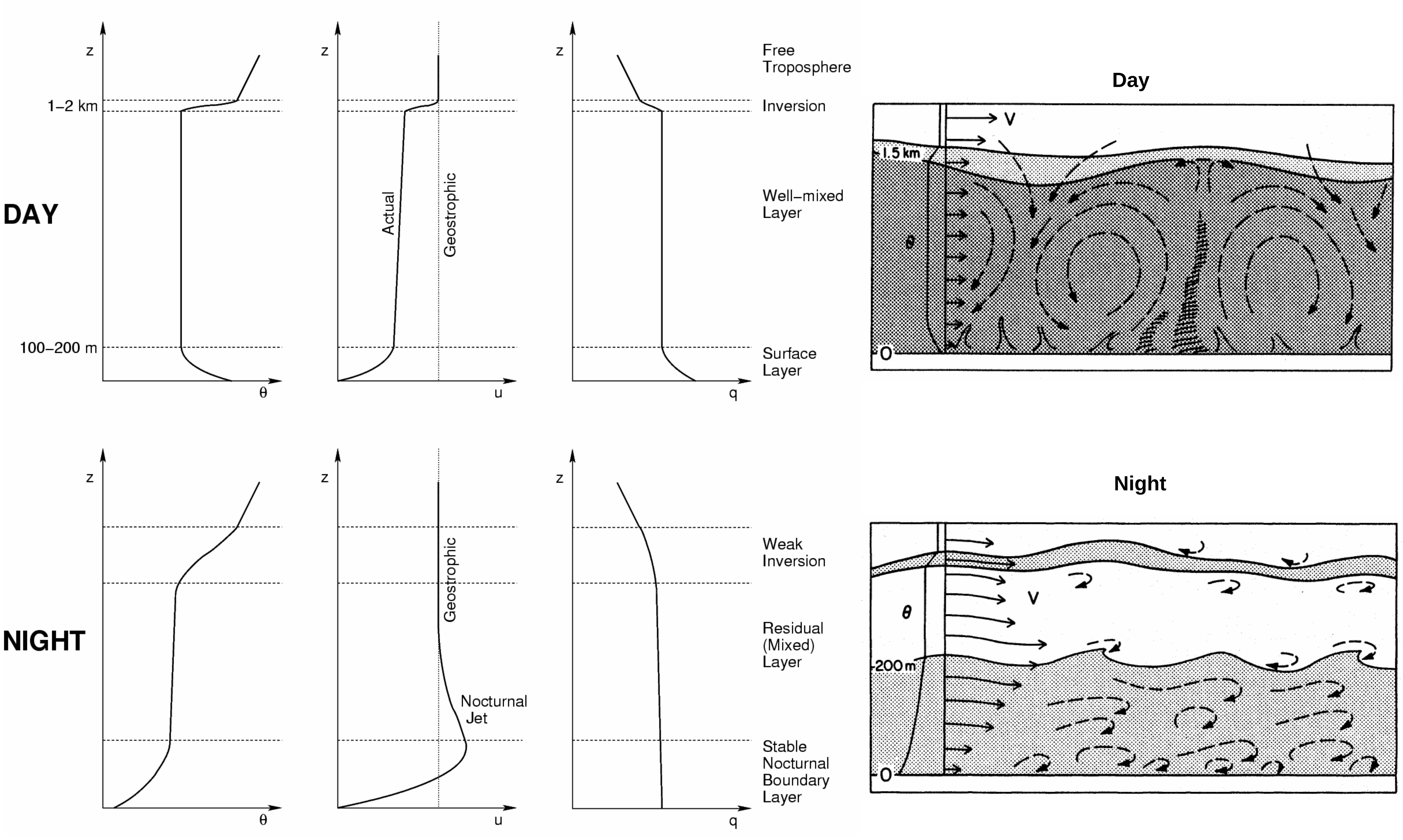
- Humans live in the boundary layer.
- Fluxes mediated here.
- Pollution dispersion.
- Local forecasting.
- Effect on the rest of the atmosphere.
- Boundary-layer clouds are very important for climate.

3. Structure of ABL over a diurnal cycle

During a clear day the boundary layer can be divided into several sublayers, as shown in Figure 1:

1. The *roughness sublayer* – this is the layer of air in which air flows around individual roughness elements (such as grass, plants, trees or buildings).
2. The *surface layer* (formerly known as the *constant flux layer*) – in this layer, typically 100 m thick (or 10% of the depth of the ABL), the winds, temperature and humidity vary rapidly with altitude, and the characteristics of turbulence are affected by the surface. Vertical fluxes of heat and momentum are approximately constant.

Figure 1. (Left) Typical profiles of potential temperature, wind and humidity over land in midlatitudes during cloudless conditions. (Right) Schematics of the typical ABL circulation and eddy structure of the ABL in the day and night (from Kaimal and Finnigan 1994).



3. The *well-mixed layer* – rising buoyant plumes from the surface layer, and associated turbulence, cause potential temperature and other quantities to be relatively constant with altitude. The earth's *rotation* becomes important in this layer, and the wind direction veers with height.
4. The *capping inversion* – on a summer's day the convective boundary layer is often capped by a temperature inversion, which inhibits mixing and confines air and pollution below it to within the boundary layer.

At night a new *stable nocturnal boundary layer* grows as air is cooled from the surface. The daytime mixed-layer remains as a *residual layer* while the capping inversion is eroded. Sometimes the ABL is difficult to define; in the vicinity of fronts there is no obvious capping inversion and the ABL structure is more a response to synoptic forcing.

4. Surface energy balance (Recap of material from MT23E)

The surface energy balance drives the diurnal variation in the ABL. For an infinitely thin "surface layer" of the ground, we have the following balance (with all terms in W m^{-2}):

$$R_n - G = H + \lambda E \,,$$

where

- R_n is the net irradiance ("net radiation") into the surface.
- G is the ground heat flux density by conduction from the surface layer of the ground into the ground beneath.
- H is the sensible heat flux density: the heating of the atmosphere by loss of energy from the surface.
- λE is the latent heat flux density: the loss of energy by evaporation (evaporation rate E has units kg m^{-2} s^{-1} and the specific latent heat of evaporation is $\lambda = 2.5\times10^6$ J kg^{-1}).

Global, diurnal mean values of these variables:

5. Bowen ratio

The nature of the ABL is determined by the balance between H and λE, which depends on surface moisture availability. A convenient parameter is the Bowen ratio, defined as

$$B = H / \lambda E.$$

It is not suitable for parameterisation schemes in numerical models but it is conceptually helpful. Typical values at midday over different surfaces are:

- **Vegetated surface**
- **Ocean**
- **Urban area**

- **Desert**

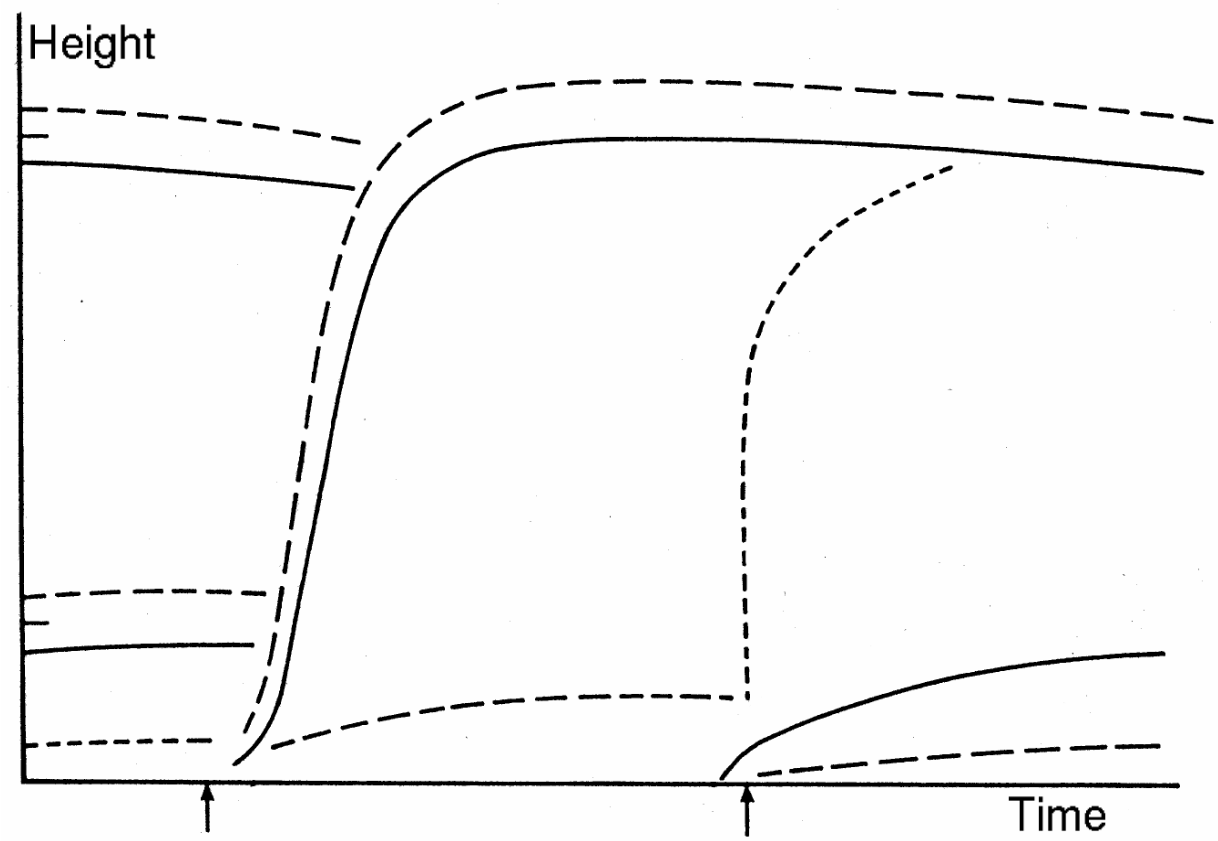


Figure 2. Schematic of the typical evolution of the boundary layer over one diurnal cycle.

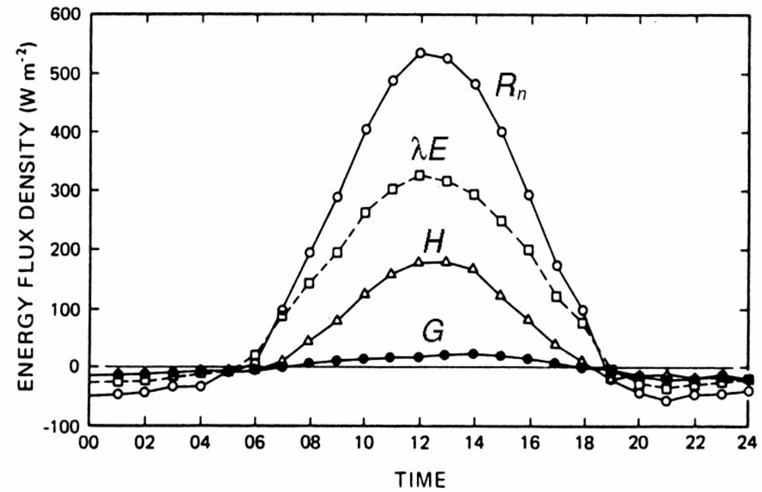


Figure 3. Diurnal cycle of the components of the surface energy budget in in cloudless conditions at a rural midlatitude site.

→ **Further reading: Stull, Chapter 1.**

B. Fundamentals of Turbulence

Conceptually, it is useful to think of the airflow within the boundary layer as consisting of three components: the mean wind, waves and turbulence. Turbulence occurs because of the shear in the mean wind, and the temperature stratification can enhance or suppress turbulence. Waves often occur in the nocturnal boundary layer, where the stable stratification supports gravity waves. The flow of air over hills is one source of waves. Turbulence promotes rapid mixing; wave motions do not. Turbulence is the source of much of the complication in modelling and measuring the boundary layer. We now consider

- The nature of turbulence
- How it affects the bulk (mean) properties in the boundary layer
- How it is modelled
- The dynamics of turbulence itself.

1. Reynolds averaging

A plot of a property (wind etc.) measured in the ABL shows high frequency variations due to turbulent motion, superimposed on low frequency variations as weather systems pass through. The easiest way to separate these two components is to think of two components: the instantaneous wind u is the sum of the mean wind $\overline{u}$ and the turbulence contribution u', such that

$$u = \overline{u} + u' , \qquad (1)$$

and similarly for θ and other variables.

→ **Sketch 1: Illustration of Reynolds averaging.**

From observations at a single point, $\overline{u}$ might be a 30-minute average, but in a numerical forecast model it could represent a horizontal spatial average over the 50-100 km represented by a single gridbox.

In meteorology, we are interested in forecasting the mean quantities such as $\overline{u}$ and $\overline{\theta}$, while u' and θ' are inherently random and the instantaneous values cannot be predicted. To determine the effect of the fluctuations on an equation we can replace each variable by the sum of its mean and its fluctuation and solve. So all occurrences of u would be replaced using (1). But by definition $\overline{u'} = 0$, so one might think that the fluctuations average to zero. However, where we have the one fluctuating quantity multiplied by another this is not the case. For example,

$$uw = (\overline{u} + u')(\overline{w} + w') = \overline{u}\,\overline{w} + \overline{u}w' + u'\overline{w} + u'w' . \qquad (2)$$

When the average value of the product *uv* is calculated, the terms $\overline{u}w'$ and $u'\overline{w}$ average out to zero, so

$$\overline{uw} = \overline{u}\,\overline{w} + \overline{u'w'} . \qquad (3)$$

We are left with an extra term, the *covariance* of *u* and *w*. What does this term mean?

2. Heat transport by turbulence

We now consider how turbulence can transport heat and momentum in the boundary layer. Consider the surface layer on a sunny day with no mean wind.

→ Sketch 2: Turbulent heat flux in a convective surface layer.

- **Excess heat energy in an individual air parcel:**

- **Transport of heat by one eddy:**

- **Mean heat transport:**

But this is just the sensible heat flux! In fact, *H* is defined at all levels in the ABL, and is the primary mechanism for heat transport through the ABL. $H(z=0)$ is the term in the surface energy budget equation.
So how does this affect the thermodynamic equation? If we neglect the small effects of radiation and molecular diffusion, and consider a boundary layer with no cloud or evaporating precipitation, then instantaneous potential temperature is conserved:

$$\frac{D\theta}{Dt} = 0 \qquad (4)$$

In the appendix to this section it is described how expanding out the advection terms using Reynolds averaging results in an extra term when we consider the rate of change of horizontal *mean* potential temperature:

$$\frac{D\overline{\theta}}{Dt} = -\frac{\partial \overline{w'\theta'}}{\partial z} = -\frac{1}{\rho C_p}\frac{\partial H}{\partial z}. \qquad (5)$$

This simply states that the rate of increase of temperature in a layer is proportional to the sensible heat entering the layer from below minus the sensible heat leaving from above.
To measure *H* we need rapid response (faster than 10 Hz) measurements of w' and θ':

- *Sonic anemometer* provides u', v' and w'.
- *Platinum resistance thermometer* provides T', which is almost identical to θ'.

From these measurements we calculate $\overline{w'\theta'}$; this is known as the *eddy correlation method*.

As an aside, it turns out that the radiation can be easily added to (5). As with sensible heat flux, the radiative heat flux can be defined at all heights in the atmosphere, so continuing the convention of the surface energy budget that R_n is positive downwards, we obtain:

$$\frac{D\overline{\theta}}{Dt} = -\frac{1}{\rho C_p}\frac{\partial (H - R_n)}{\partial z}. \qquad (6)$$

Note that in clear air the addition of R_n typically leads to a cooling of 1-3 K per day, whereas H can lead to a warming or cooling of several K per hour. However, at the boundaries of clouds the radiation term can dominate.

3. Momentum transport by turbulence

In the same way as turbulence transports heat it transports momentum. Consider the case when the mean wind is directed in the x direction.

→ Sketch 3: Turbulent momentum flux.

- **Excess horizontal momentum in an individual air parcel:** $\rho u'$
- **Vertical momentum transport by one eddy:** $\rho u'w'$
- **Mean momentum flux:** $\rho \overline{u'w'}$ (N m^{-2})

Because wind speed invariably increases with height in the surface layer, $\overline{u'w'}$ is invariably negative. We therefore often use the *Reynolds stress* $\tau_{xz} = -\rho \overline{u'w'}$. The surface value of τ_{xz} is the drag (force per unit area) of the atmosphere on rough elements of the surface, but is also the drag that roughness elements exert on the wind. It is therefore crucial to represent in weather models.

We can see how this affects the momentum equation by performing the same Reynolds averaging (see the appendix). Neglecting the small effect of molecular diffusion, the x-momentum equation for a small parcel of air is given by

$$\frac{Du}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + fv \,. \qquad (7)$$

After Reynolds averaging an extra term is introduced representing the frictional force per unit mass due to turbulence

$$\frac{D\overline{u}}{Dt} = -\frac{1}{\rho}\frac{\partial \overline{p}}{\partial x} + f\overline{v} - \frac{\partial \overline{u'w'}}{\partial z} \,, \qquad (8)$$

and similarly in the other momentum equations.

4. The turbulence closure problem

What do we do with the extra terms that have appeared in the thermodynamic and momentum equations? We might attempt to find them by writing predictive equations for them, i.e. an equation for $\partial \overline{u'w'} / \partial t$. However, when we do this (using Reynolds averaging) we find that the equation contains third-order terms such as $\overline{u'v'w'}$. A predictive equation for this contains even nastier terms that we don't know what to do with. This problem is known as the *turbulence closure problem*.

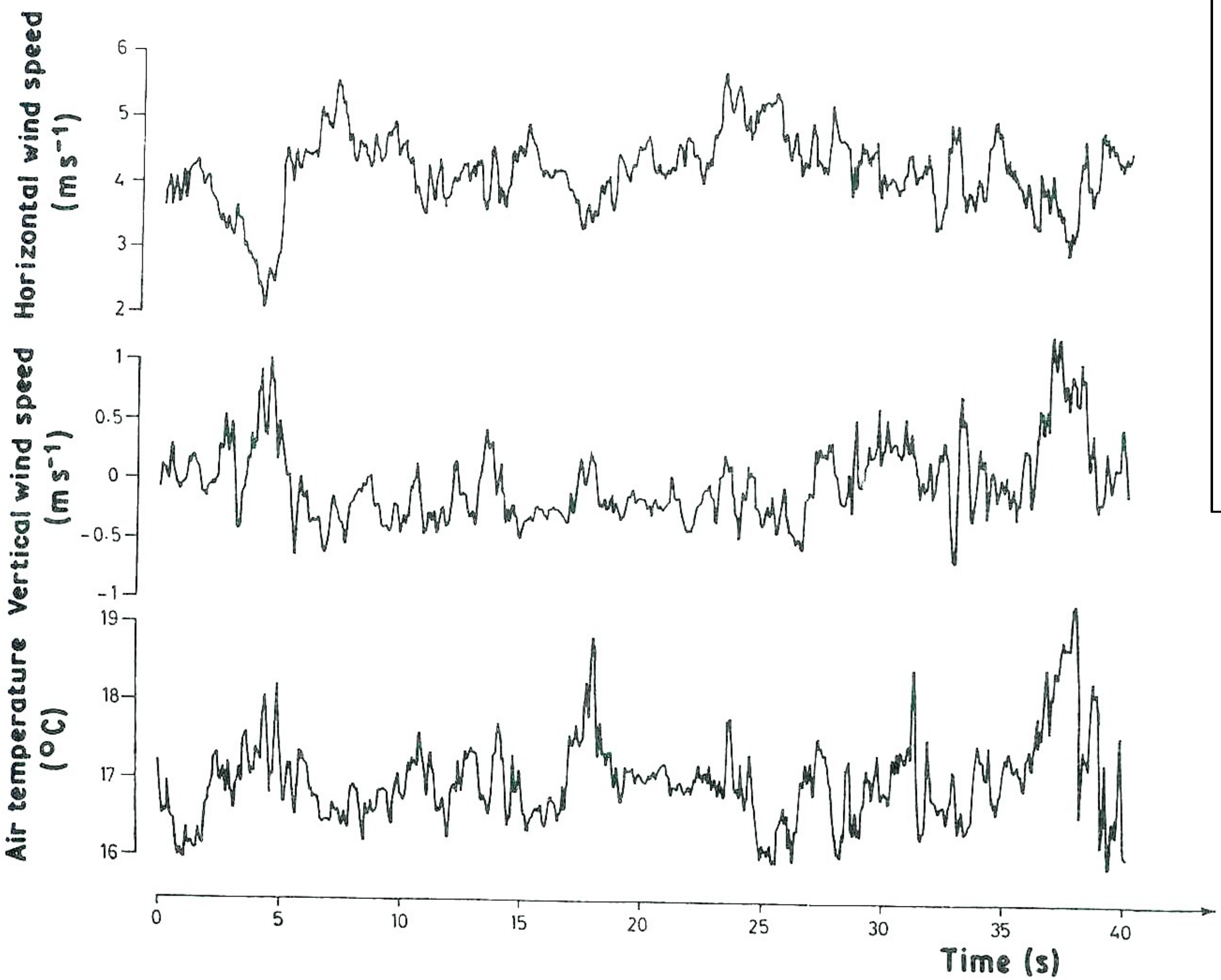


Figure 1. A short section of horizontal wind speed, vertical wind speed and temperature at a height of two metres at the Reading University Atmospheric Observatory. Note the positive correlations between vertical wind speed and temperature, and the negative correlations between horizontal and vertical wind speeds.

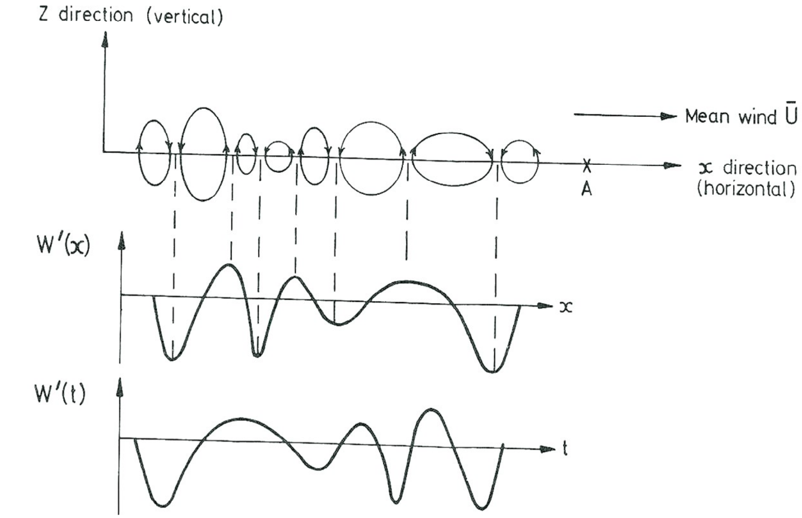


Figure 2. Idealised vertical cross-section through a series of turbulent eddies advecting past an observer at point A.

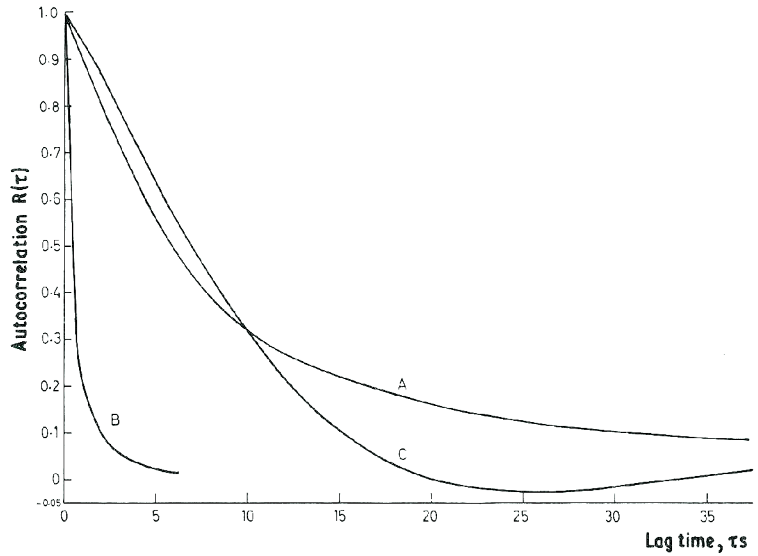


Figure 3: Autocorrelation functions for data observed at z=2m in the surface layer. Curves A and B are for horizontal wind speed u' and vertical wind speed w' over flat ground. Curve C is the autocorrelation of horizontal windspeed shown in Fig. 1.

One way to get around the turbulence closure problem is to retain terms up to a particular order and approximate the rest. In the following section we will see an example of a first order closure, where the first moments ($\overline{u}$) are retained and the second moments ($\overline{u'w'}$) are approximated. Closure approximations can be divided into *local schemes*, where the unknown quantities are parameterised in terms of local known quantities (such as the mean wind shear and temperature gradient), and *non-local schemes*, where the unknown quantities depend on the boundary layer properties over a larger region of space.

5. Local first order closure: K theory

We have already seen how the gradient of a quantity tends to determine the direction of the turbulent flux, with heat tending to flow from hot to cold. A simple approximation is to assume that vertical transport is proportional to the gradient of the mean. So for temperature flux:

- **Temperature flux:**

- **Momentum flux:**

In these equations, K_h and K_m are the *eddy diffusivities* (in m^2 s^{-1}) for heat and momentum, respectively. This approach is known as *K-theory* or the *flux-gradient* method. Note that other scalars tend to have the same eddy diffusivity as heat. The treatment of turbulence in this way is exactly analogous to the way in which molecular diffusion works. If we consider how this would appear in the thermodynamic equation, then assuming K_h to be constant with height we have:

$$\frac{D\overline{\theta}}{Dt} = -\frac{\partial \overline{w'\theta'}}{\partial z} = K_h \frac{\partial^2 \overline{\theta}}{\partial z^2}. \qquad (9)$$

This term has the same form and hence the same effect as molecular diffusion, in that it tries to mix the profile until it is uniform (i.e. $\overline{u}$ and $\overline{\theta}$ independent of z). In a convective boundary layer this is realised: the profile becomes *well mixed*. However, K_h is typically around 10^6 times larger than the molecular thermal diffusion coefficient κ, so turbulence is far more efficient at mixing.
However, we still need to know what K_h and K_m are. Various approaches have been taken and will be covered in future lectures; Ekman considered the eddy diffusivity to be constant through the ABL, but in the surface layer we can relate it to the distance from the surface and in the rest of the boundary layer it can be related to stability.

→ Further reading: Holton p116-121.

Appendix to B: Reynolds averaging of the advection terms (non-examinable)

The Lagranian derivative of a quantity *a* (which could be *u*, θ etc.) can be written as

$$\frac{Da}{Dt} = \frac{\partial a}{\partial t} + u\frac{\partial a}{\partial x} + v\frac{\partial a}{\partial y} + w\frac{\partial a}{\partial z}. \quad (10)$$

By continuity,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \quad (11)$$

so we can add a multiple (*a*) of the above three terms to the equations of motion without changing anything, as they sum to zero:

$$\frac{Da}{Dt} = \frac{\partial a}{\partial t} + u\frac{\partial a}{\partial x} + v\frac{\partial a}{\partial y} + w\frac{\partial a}{\partial z} + a\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right). \quad (12)$$

By the chain rule for differentiation this may be rewritten as

$$\frac{Da}{Dt} = \frac{\partial a}{\partial t} + \frac{\partial ua}{\partial x} + \frac{\partial va}{\partial y} + \frac{\partial wa}{\partial z}. \quad (13)$$

We then apply Reynolds decomposition and rewrite *a*, *u*, *v* and *w* as the sum of the mean and a fluctuating part. When averaged we obtain

$$\overline{\frac{Da}{Dt}} = \frac{\partial \overline{a}}{\partial t} + \frac{\partial \overline{u}\overline{a}}{\partial x} + \frac{\partial \overline{v}\overline{a}}{\partial y} + \frac{\partial \overline{w}\overline{a}}{\partial z} + \frac{\partial \overline{u'a'}}{\partial x} + \frac{\partial \overline{v'a'}}{\partial y} + \frac{\partial \overline{w'a'}}{\partial z}. \quad (14)$$

Note also that the operation of differentiating a quantity does not alter the fact that it averages out to zero over a long enough period. We then expand the derivatives on the left hand side of the equation using the chain rule and use the continuity equation to cancel terms in the reverse of the step used to go from (10) to (13) to obtain

$$\overline{\frac{Da}{Dt}} = \frac{\partial \overline{a}}{\partial t} + \overline{u}\frac{\partial \overline{a}}{\partial x} + \overline{v}\frac{\partial \overline{a}}{\partial y} + \overline{w}\frac{\partial \overline{a}}{\partial z} + \frac{\partial \overline{u'a'}}{\partial x} + \frac{\partial \overline{v'a'}}{\partial y} + \frac{\partial \overline{w'a'}}{\partial z}. \quad (15)$$

As with almost all other properties of the atmosphere (temperature, mixing ratio, wind speed, pressure etc.), the turbulent flux terms have a *much larger gradient in the vertical than the horizontal*. We are therefore justified in neglecting all but the last turbulent flux term, to leave

$$\begin{aligned} \overline{\frac{Da}{Dt}} &= \frac{D\overline{a}}{Dt} + \frac{\partial \overline{w'a'}}{\partial z} \\ &= \frac{\partial \overline{a}}{\partial t} + \overline{u}\frac{\partial \overline{a}}{\partial x} + \overline{v}\frac{\partial \overline{a}}{\partial y} + \overline{w}\frac{\partial \overline{a}}{\partial z} + \frac{\partial \overline{w'a'}}{\partial z}. \end{aligned} \quad (16)$$

Comparing (10) and (16) we see that if we are to apply the Navier-Stokes equations to an average quantity rather than an instantaneous one at a point in space, we must introduce an extra term due to the vertical transport of that quantity by turbulent eddies.

C. Statistical description of turbulence

1. General statistics

For any scalar variable θ, the concept of *Reynolds Averaging* (see MT23E) allows us to partition a variation into mean and fluctuating parts: $\theta_i = \overline{\theta} + \theta_i'$. Here, θ can be considered as a function of *space* (observed at a fixed time), or as a function of *time* (observed at a fixed spatial location). The *variance* of the fluctuation is measured by

$$\sigma^2 = \overline{\theta'^2} \quad (1)$$

A measure of "gustiness" in the wind field is given by the *turbulence intensity*, defined as $\sigma_u / \overline{u}$ for the streamwise component of the wind, *u*. Turbulence intensity is typically 10-30% in the surface layer. In practice, field data used to measure the properties of turbulence usually consist of *N* observations sampled at equal time intervals (Δ_t) over a period of time $N\Delta_t$. The *mean* and *variance* statistics are estimated using

$$\hat{\overline{\theta}} = \frac{1}{N}\sum_{i=1}^{N}\theta_i \; ; \quad (2)$$

$$\hat{\overline{\sigma}}^2 = \frac{1}{N}\sum_{i=1}^{N}\left(\theta_i - \hat{\overline{\theta}}\right)^2 \equiv \frac{1}{N}\sum \hat{\theta}_i'^2 \; , \quad (3)$$

where the caret (or "hat") indicates an *estimate* of something that is derived from the raw measurements θ_i. The estimation of an eddy correlation quantity, such as $\overline{w'\theta'}$ is obtained from

$$\overline{w'\theta'} = \frac{1}{N}\sum_{i=1}^{N}\hat{w}_i'\hat{\theta}_i' \quad (4)$$

This is just the 'multivariate' version of (3), known as the *covariance*.

If advection or other large-scale changes affect the record, then this can lead to misleading measures of turbulent eddy statistics, i.e. slower, low frequency changes are attributed as being due to turbulence when actually they are not. Such effects can be removed by subtracting a *trend* from the original series to define the turbulent variations, instead of simply subtracting the mean defined by (2). It is often sufficient to assume a *linear trend:*

$$\overline{\theta}_i \equiv \overline{\theta}\left(t = i\Delta_t\right) = \alpha + \beta t \quad (5)$$

The unknown coefficients α, β are easily estimated using *ordinary least-squares fitting*. The process of subtracting the estimated trend from the original data is known as *de-trending*.

The covariance between horizontal and vertical wind components, $\overline{u'w'}$, allows us to calculate the momentum flux, *τ*. We can normalise this quantity by dividing by the combined variance in *u* and *w*, and thus define a correlation coefficient

$$R_{uw} = \frac{\overline{u'w'}}{\sigma_u \sigma_w} \tag{6}$$

This can be thought of as the *turbulent efficiency*, i.e. what fraction of the turbulent "sloshing about" is actually resulting in the turbulent diffusion of momentum. A typical value in the surface layer is ~ –0.3.

2. Structure, time-scales and length-scales

As well as calculating the correlation between two variables, it is instructive to look at the *lag auto-correlation*, or the correlation of a variable with itself at later timesteps. For instance, consider the *u* component of the wind

$$R_u(\tau) = \frac{\overline{u'(t)u'(t+\tau)}}{\sigma_u{}^2} \tag{7}$$

Hence $R(\tau)$=1 at τ=0. If the turbulence is relatively *homogeneous* (meaning that its properties are not varying in space), a time lag T is equivalent to a spatial lag X (measured in the direction of the mean wind) equal to $\bar{u}T$. If these conditions are valid then

$$R(X) = R(T) \qquad when \qquad X = \bar{u}T \tag{8}$$

→ See Fig. 3 of section B for a typical auto-correlation versus lag time in the surface layer for u, v and w.

The way in which $R(\tau)$ varies with lag is related to the *size distribution* of eddies. Large eddies cause slower variations in the time series, and thus the auto-correlation will decrease more slowly with lag than for a time series dominated by smaller eddies (compare autocorrelation functions for vertical wind-speed and horizontal wind-speed in Fig. 3 of section B – what does this tell us?). Hence, a simple measure of 'size' is given by the integral timescale L_T , defined as

$$L_T = \int_0^\infty R(\tau)d\tau \tag{9}$$

From Taylor's frozen turbulence hypothesis, $L_X \cong \bar{u}L_T$. In the case of the vector wind field, one-dimensional covariance functions can be defined for all three components. In relation to surface momentum exchange, the auto-correlations for *u* and *w* are of greatest interest.

→ Further reading: Panofsky and Dutton; Kaimal and Finnigan chapter 2 and appendices; Ibbetson, A. 1981, in "Dynamical Meteorology: An Introductory Selection", ed. B.W. Atkinson, Methuen

D. The dynamics of turbulence and the Richardson Number

A key question is how turbulent mixing is affected by the stability of the atmosphere. On a sunny day the surface is heated and the boundary layer becomes unstable, promoting convective plumes and turbulent mixing. On a clear night the surface is cooled and the boundary layer becomes stable, suppressing vertical movement of air parcels (up or down). This will obviously affect turbulent fluxes (e.g. via eddy diffusivities) but how? We need to quantify what encourages turbulence and what suppresses it.

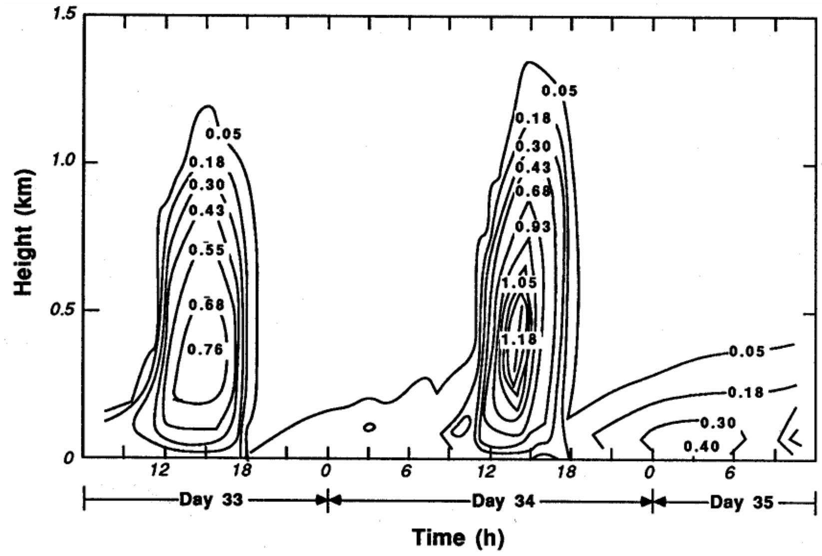


Figure 1. Diurnal evolution of turbulent kinetic energy per unit mass, e (m² s⁻²), simulated for the Wangara experiment. From Yamada and Mellor (1975).

1. Turbulent kinetic energy

We consider the *turbulent kinetic energy* (TKE) per unit mass, e, defined as

$$e = \qquad\qquad\qquad\qquad \mathrm{m^2\,s^{-2}\ or\ J\,kg^{-1}}$$

This represents the kinetic energy in turbulent motions. Using Reynolds averaging leads to an equation for the rate of change of e with time:

$$\frac{\partial e}{\partial t} = \text{shear production} + \text{buoyancy production} + \text{transport} - \text{dissipation } \varepsilon,$$

where for mean wind in the x direction:

- **Shear production** =

- **Buoyancy production** =

"Transport" includes advection and turbulent fluxes but its exact form is not relevant here.

2. Richardson number

From the TKE equation, we can see that whether turbulence grows or decays depends on the relative magnitude of the shear and buoyancy terms. A very useful quantity is the *Richardson Number*:

- **Flux Richardson Number** $R_f = \dfrac{\frac{g}{\overline{\theta}}\overline{w'\theta'}}{\overline{u'w'}\frac{\partial \overline{u}}{\partial z}} =$

Hence we can write the TKE equation as

$$\frac{\partial e}{\partial t} = \text{shear production} \times (1 - R_f) - \varepsilon\,,$$

A more common form is derived by applying first order closure:

$$\overline{u'\theta'} = -K_h \frac{\partial \overline{\theta}}{\partial z} \qquad \overline{u'w'} = -K_m \frac{\partial \overline{u}}{\partial z},$$

which leads to

- **Gradient Richardson Number** $R_i =$

If this is applied over a finite layer it is referred to as Bulk Richardson Number, and if the mean wind is coming from any direction (rather than just the *x* direction) it is given by

- **Bulk Richardson Number** $R_i = \dfrac{\frac{g}{\overline{\theta}}\frac{\Delta\overline{\theta}}{\Delta z}}{\left(\frac{\Delta\overline{u}}{\Delta z}\right)^2 + \left(\frac{\Delta\overline{v}}{\Delta z}\right)^2}$.

The Richardson Number is useful because we can diagnose the onset of turbulence in statically stable but sheared flow. Clearly we have convective instability for $R_i < 0$, but theoretical and experimental studies show

- **Non-turbulent flow** becomes turbulent when R_i drops below the critical value R_{ic} of around 0.25.
- **Turbulent flow** becomes non-turbulent when R_i becomes larger than around 1.

Hence there is a *hysteresis*. When R_i drops below 0.25, the first appearance of turbulence is in the form of *Kelvin-Helmholtz waves*.

3. The dissipation term

The dissipation term in the TKE equation, ε (Greek letter "epsilon"), represents the fact that over time turbulent energy is dissipated into heat. This process arises due to the non-linear nature of the equations of motion and involves a *cascade of energy* to smaller scales: large eddies continually break up into smaller ones until all energy is in the form of molecular motions only, i.e. heat. The amount of heat produced is so small that it is rarely included as a source term in the thermodynamic equation. Richardson described this effect in a rhyme:

Big whorls have little whorls
That feed on their velocity,
And little whorls have lesser whorls
And so on to viscosity.

→ Further reading: Stull p151-165.
→ More on Kelvin-Helmholtz instability: http://www-sccm.stanford.edu/Students/witting/kh.html
http://www.enseeiht.fr/hmf/travaux/CD0001/travaux/optmfn/hi/01pa/hyb72/kh/kh_theo.htm

E. Micrometeorology of the surface layer

Despite the effect of a complex surface on turbulent flow, the theory of the surface layer is more developed than for the boundary layer as a whole because the presence of the ground limits the size of the eddies, and because a large number of ground-based measurement campaigns have been carried out using instrumented masts. This section presents the theory of determining the profile of $\overline{u}$ and $\overline{\theta}$ in the surface layer.

To simplify matters we treat *u* to be in the direction of the mean wind and assume that turbulent fluxes of heat and momentum are constant in the surface layer. In the case of a convective boundary layer, the sensible heat from the surface is distributed evenly through the whole boundary layer, so the sensible heat flux at the top of the surface layer (the lowest 10% of the boundary layer) is typically 90% of the surface value so it is reasonable to assume it to be constant through the surface layer. Similar reasoning applies to momentum. We represent the constant momentum flux $\overline{u'w'}$ with the *friction velocity* u_*, defined as

$$u_* = \sqrt{-\overline{u'w'}}\,. \qquad (1)$$

As $\overline{u'w'}$ is always negative in the surface layer, u_* is always positive. It is typically around 0.2 m s^{-1} in the day.

1. Mixing length theory

A simple way of thinking about turbulent transfer is in terms of *mixing length*, a concept introduced by Prandtl. The mixing length l_m is the typical vertical depth over which parcels of air move before mixing with their surroundings. It can be thought of as the radius of a typical eddy. While this is a crude model of turbulence, it is helpful in suggesting what will be proportional to what, but leaving it to experiments to determine the constants of proportionality.

→ Sketch 1: How mixing length is related to u' and w'.

This leads to the following relationship between friction velocity, mixing length and the shear of the mean wind.

$$u_* = l_m \frac{d\overline{u}}{dz}\,. \qquad (2)$$

As u_* is constant with height, if we know l_m then we can determine the wind shear and hence the wind speed.

2. Neutral conditions: the log profile

Consider first the simplest case of the wind profile in a neutral surface layer. In this situation the temperature variables will not play an active role, as indicated by the fact that the Richardson number will be small (indicating that shear production of turbulence is much larger than buoyant production or suppression). So what is the mixing length likely to depend on? It turns out that it is dependent *only* on distance from the surface, *z*. Graphically this is represented in the left-most panel of Fig. 1, showing the *typical* eddy size to increase linearly with height. Substituting $l_m = kz$ (where *k* is a dimensionless constant) into (2) we find that the wind shear is inversely proportional to *z*:

$$\frac{d\overline{u}}{dz} = \frac{u_*}{kz}. \qquad (3)$$

We know that wind speed must fall to zero at some height we will call z_0. Integrating this expression:

$$\int_0^{\overline{u}} d\overline{u} = \frac{u_*}{k} \int_{z_0}^{z} \frac{dz}{z}$$

yields the *log wind profile:*

$$\boxed{\overline{u} = \frac{u_*}{k} \ln\left(\frac{z}{z_0}\right)} \qquad (4a)$$

As you will find in experiment 1, this is an excellent model for the wind profile near the ground in almost all conditions. In this equation

- k is *von Karman's constant* and has a value, derived from observations, of around 0.4. Amazingly it is the same for all turbulent fluids. Note that it is sometimes written as κ (the Greek letter "kappa").
- z_0 is the *roughness length,* defined as the height where the wind according to the log law falls to zero. In fact z_0 lies within the roughness sub-layer where $\overline{u}$ deviates from the log law. It represents the bulk effects of roughness elements in the surface layer and very approximately has a value around 0.1 times the height of the roughness elements.

Over an array of densely packed objects (e.g. a forest, a crop of wheat or a city), we have to introduce an offset in height to allow for the upward displacement of the flow by the surface objects. Thus the log-law becomes:

$$\overline{u} = \frac{u_*}{k} \ln\left(\frac{z-d}{z_0}\right) \qquad (4b)$$

Where d is the *displacement height*. A "rule of thumb" is that d is around two-thirds of the canopy height.

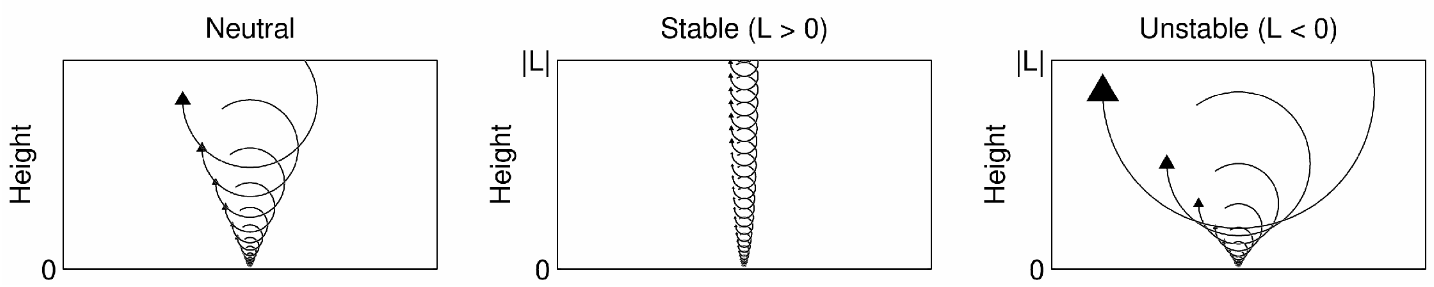


Figure 1. Schematic indicating the typical size of the eddies that are responsible for the mixing as a function of height and Obukhov length L, for neutral, stable and unstable conditions. The Businger-Dyer relations were used, and eddies drawn with a radius equal to the mixing length l_m.

3. Non-neutral conditions: Monin-Obukhov scaling of the surface layer

Most surface layers are either stable or unstable. The effect of buoyancy is to increase turbulence and hence to change the diffusivity for a particular value of momentum flux and height. Conceptually it is easiest to think of this as changing the typical size of the eddies, as illustrated in Fig. 1 (don't worry about the meaning of L for the moment). We modify our definition of the mixing length to reflect this:

$$l_m = kz / \phi_m \, , \qquad (5)$$

where ϕ_m is a dimensionless parameter related to stability.

In the absence of a physical theory of turbulence, Monin and Obukhov (1954) used *dimensional analysis* to help determine ϕ_m. In this procedure one lists the variables that one thinks the solution will depend on, and combines them in such a way (using multiplication, division etc.) that the combination has the correct units. In our case we want to know the value of the dimensionless parameter ϕ_m.

- **What are the parameters that ϕ_m is likely to depend on?**

The only dimensionless form is $\zeta = z / L$, where ζ is referred to as the *Monin-Obukhov stability parameter*, and the *Obukhov length* is defined as

- **Obukhov length** $L =$

Note that it is a function of *surface variables only*. ϕ_m is a function of $\zeta = z / L$ *only*. This function must be determined by experiment but should be universally applicable. Substitution of (5) into (2) yields

$$\frac{d\overline{u}}{dz} = \frac{u_*}{kz} \phi_m \left(\frac{z}{L} \right) . \qquad (6)$$

4. Physical interpretation of the Monin-Obukhov stability parameter

We can interpret $\zeta = z/L$ physically. In the turbulent kinetic energy equation we have

- **Buoyancy production term** $= \dfrac{g}{\overline{\theta}}\overline{w'\theta'} = \dfrac{g}{\overline{\theta}}\dfrac{H}{\rho C_p}$.

So the grouping of $\overline{w'\theta'}$ and $g/\overline{\theta}$ is no accident: this term directly affects turbulence. Also in the TKE equation:

- **Shear production term** $= -\overline{u'w'}\dfrac{\partial\overline{u}}{\partial z} = +u_*^2\dfrac{\partial\overline{u}}{\partial z}$ and in a neutral surface layer this term equals $\dfrac{u_*^3}{kz}$.

Hence

- $\zeta = z/L =$ buoyancy suppression ÷ shear production in *equivalent neutral conditions*.

From this definition it is no surprise that z/L is uniquely related to gradient Richardson number, and in unstable conditions (i.e. when $R_i < 0$), it turns out that $R_i = z/L$. $|L|$ is the height above the ground where the buoyancy and shear production of TKE are of equal magnitude; below this height shear dominates and above it buoyancy dominates.

Interpretation of the value of *z/L*:

5. The Monin-Obukhov functions

From experiments in very homogeneous conditions, the following *Businger-Dyer* relationships have been found between ϕ_m and z/L (Dyer 1974, Garratt 1992):

- **Stable conditions** ($z/L > 0$): $$\phi_h = \phi_m = 1 + 5\frac{z}{L} \qquad (7)$$
- **Unstable conditions** ($z/L < 0$): $$\phi_h = \phi_m^2 = \left(1 - 16\frac{z}{L}\right)^{-1/2} \qquad (8)$$

Note that we should generally restrict the application of these relationships to $z < |L|$.

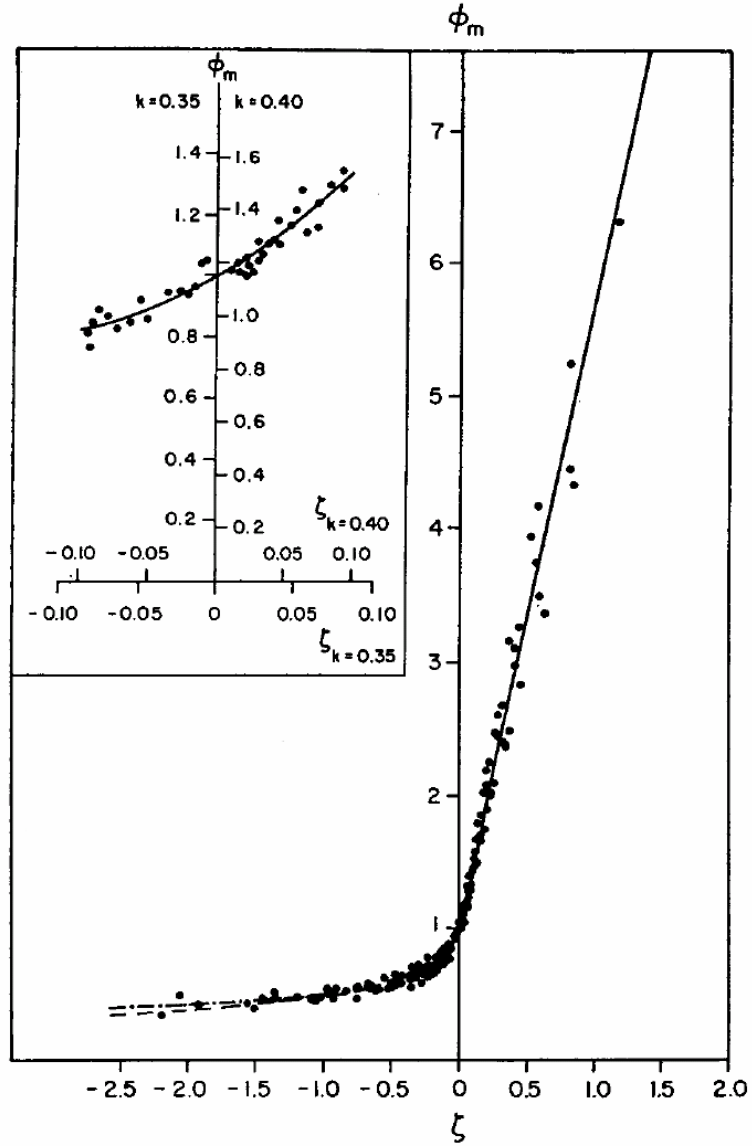
There are also some useful relationships with gradient (or bulk) Richardson Number, which is easier to measure than *L* because it depends only on gradients of mean quantities:

- **Stable conditions** ($z/L > 0$; $0 \le R_i < 0.2$): $$z/L = \frac{R_i}{1 - 5R_i} \qquad (9)$$
- **Unstable conditions** ($z/L < 0$; $R_i < 0$): $$z/L = R_i \qquad (10)$$

From ϕ_m we can calculate $\bar{u}$ in stable conditions:

→ **Revision tip: You don't need to memorise (7)-(10).**

Figure 2. Observations of ϕ_m versus the Monin-Obukhov stability parameter $\zeta = z/L$ by Businger et al. (1971, J. Atmos. Sci., 28, 181-189).



Alternatively this can be written as

$$\bar{u} = \frac{u_*}{k}\left[\ln\frac{z}{z_0} - \psi_m\left(\frac{z}{L}\right)\right],$$

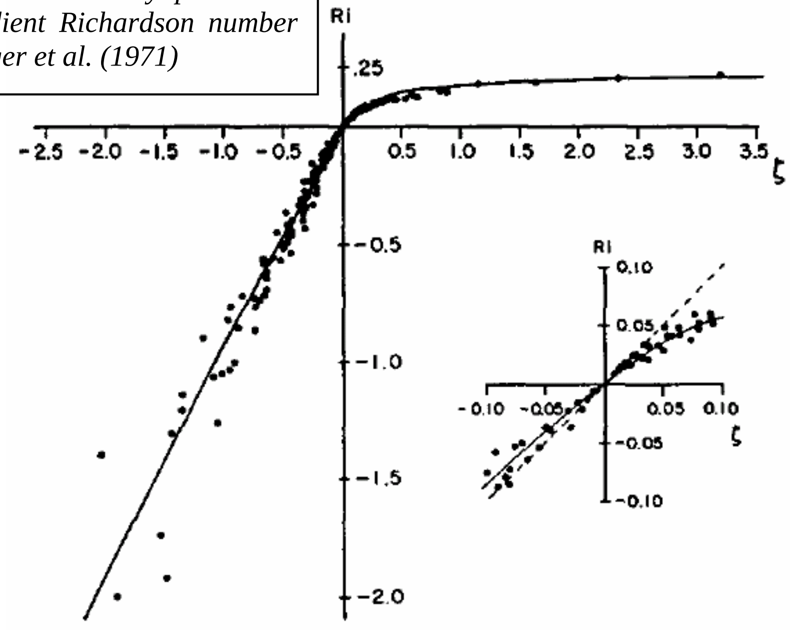
where $\psi_m(z/L) = -5z/L$ in stable conditions. In *unstable* conditions we still use this form, but the integration has a more complex result:

$$\psi_m\left(\frac{z}{L}\right) = 2\ln\left(\frac{1+X}{2}\right) + \ln\left(\frac{1+X^2}{2}\right) - 2\tan^{-1}(X) + \frac{\pi}{2}, \qquad \text{where} \quad X = \left(1-16\frac{z}{L}\right)^{1/4}.$$

(Don't memorize this!)

→ **Sketch 1: How surface buoyancy affects the wind profile.**

Figure 3. Observations of the Monin-Obukhov stability parameter versus gradient Richardson number from Businger et al. (1971)



6. The temperature profile in non-neutral conditions

So far the discussion has been with regard to turbulent momentum transport, but for other parameters, particular heat, we use the dimensionless parameter ϕ_h, which turns out to be equal to ϕ_m in stable conditions but equal to ϕ_m^2 in unstable conditions. Defining a turbulent temperature scale $\theta_* = -\overline{w'\theta'}/u_*$, we can write an analogous expression as (8) but for temperature:

$$\frac{d\bar{\theta}}{dz} = \frac{\theta_*}{kz}\phi_h\left(\frac{z}{L}\right). \qquad (11)$$

The solution is

$$\bar{\theta} = \bar{\theta}_0 + \frac{\theta_*}{k}\left[\ln\frac{z}{z_T} - \psi_h\left(\frac{z}{L}\right)\right], \qquad (12)$$

where $\bar{\theta}_0$ is the surface temperature and z_T is the roughness length for temperature, and is usually less than the aerodynamic roughness length z_0. In stable conditions, $\psi_h = \psi_m = -5z/L$ and in unstable conditions $\psi_h = 2\ln\left[(1-X^2)/2\right]$.

7. Parameterising K in numerical models (non-examinable)

How does this relate to K theory, and how mixing is represented in numerical models? Recall that the eddy diffusivity for momentum is defined such that

$$\overline{u'w'} = -K_m \frac{\partial \overline{u}}{\partial z}. \qquad (13)$$

Substituting in (1) and (2) leads to

$$K_m = l_m^2 \frac{\partial \overline{u}}{\partial z} \qquad (14)$$

In the surface layer, use of (2) and (5) enables this to be expressed as

$$K_m = u_* l_m = u_* kz / \phi_m \qquad (15)$$

Hence in neutral conditions $(\phi_m = 1)$ the eddy diffusivity is proportional to distance above the surface. In numerical models we need to blend smoothly between the surface layer and the rest of the boundary layer. In the Met Office model, the mixing length in (14) is split into a *neutral mixing length* λ_m and a stability function $f_m(R_i)$:

$$K_m = \lambda_m^2 f_m(R_i) \frac{\partial \overline{u}}{\partial z} \qquad (16)$$

The neutral mixing length can be formulated to vary smoothly between *kz* in the surface layer and a constant value in the rest of the boundary layer above:

$$\frac{1}{\lambda_m} = \frac{1}{kz} + \frac{1}{\lambda_0}. \qquad (17)$$

Typically λ_0 is constant at around 50 m, or is some fraction of the boundary layer depth. A number of formulations for $f_m(R_i)$ have been suggested, but one that arises naturally in stable conditions by combining (7) and (9) results in (16) becoming

$$K_m = \lambda_m^2 (1 - 5R_i)^2 \frac{\partial \overline{u}}{\partial z} \qquad (18)$$

This would be valid for $0 < R_i < 0.2$ and above 0.2, K_m would be set to zero (no mixing).

K theory works well in near-neutral and stable situations, but in convective boundary layers, $\overline{u}$ and $\overline{\theta}$ are well-mixed, i.e. constant with height. From (14), zero gradient implies zero turbulent flux, but the heat flux in particular may be very high. For this reason a *non-local scheme* is applied in the Met Office model in unstable conditions, in which K_m is not dependent on local gradients, but rather on the depth of the unstable layer and the sensible heat flux at the bottom of the layer.

→ Further reading: Stull p180-184, Arya p92-94, Kaimal & Finnigan p11-19.

F. Drag Law

In Numerical Weather Prediction and in engineering, it is often convenient to parameterise the surface drag in terms of the wind speed u_r at some reference level z_r, such as 10 m. The corresponding *drag law* takes the form

$$\tau = \rho u_*^2 = \rho C_D u_r^2, \qquad (1)$$

where C_D is a dimensionless *drag-coefficient*. In the case of a near-neutral ABL, $u_r = \left(u_* / k\right)\ln\left(z_r / z_o\right)$, so that

$$C_D \cong \frac{k^2}{[\ln\left(z_r / z_o\right)]^2} \equiv \left[\frac{u_*}{\overline{u}(z_r)}\right]^2. \qquad (2)$$

Hence the neutral surface layer drag coefficient depends only on the reference height and z_o. In non-neutral conditions, a more complex expression is required.

G. Micrometeorological methods for measuring surface fluxes

Fluxes of heat, momentum and water vapour within the surface layer can be estimated *either* directly from measurements of the turbulent fluctuations themselves *or* indirectly from measurements of mean variables. The following is a brief summary of principles; for further details refer to textbooks and practical classes.

1. Eddy Correlation method

In the case of H and τ, fast response anemometer and thermometer instruments can be used to measure eddy correlations directly. Care must be taken to 'match' the response of either the instruments themselves, or numerical filters chosen to separate long and short time-scale fluctuations in the data. Directional anemometers are required to define fluctuations *along* the direction of the mean wind. It is difficult to use the same approach for estimating E. Instead, E is often calculated as a residual term in the SEB equation, assuming reliable measurements of R_n and G.

To obtain relatively accurate estimates of $\overline{w'\theta'}$, it is necessary to measure at least 200 data pairs. More like 800 data pairs are required in the case of $\overline{w'u'}$. The sampling period should be less than 1 hour (30 minutes is typical), otherwise the effect of diurnal variation is to produce misleading results due to trends. This means that the sampling rate should not be less than about 0.5 Hz.

2. Profile Methods

In Part E, the modified profile laws were introduced:

$$\overline{u} = \frac{u_*}{k}\left[\ln\frac{z}{z_0} - \psi_m\left(\frac{z}{L}\right)\right] \qquad \text{and} \qquad \overline{\theta} = \overline{\theta}_0 + \frac{\theta_*}{k}\left[\ln\frac{z}{z_T} - \psi_h\left(\frac{z}{L}\right)\right],$$

suggesting that generally a vertical profile is of the form

$$q(z) = A + B\left\{\ln(Z) - \psi_q\left(\frac{Z}{L}\right)\right\} \qquad (1)$$

where $Z = (z-d)$ and A and B are constants that can be estimated given measurements of q at two or more levels, provided L is known. For wind speed, $B = u_*/k$ while for temperature, $B = -T_*/k$. Hence the heat flux $H = \rho c_p u_* T_*$ is easily obtained from the slopes of the fitted profiles.

An obvious difficulty arises because L is initially unknown. Two solution methods are:

a) In unstable conditions, use the observations to estimate the Richardson Number Ri. Then use $z/L = Ri$ to estimate L. The values of ψ can thus be calculated at the various sampling heights.

b) An iterative approach starts by assuming neutral conditions ($L = \infty$) so that the slope parameters are obtained assuming log profiles. This provides preliminary estimates of u_* and T_*, from which L can be calculated using

$$L = -\frac{T_0 u_*^2}{kgT_*} \qquad (2)$$

This value is used to recalculate the modified height coordinate in (1), so that revised estimates of the slope parameters can be obtained. The process is then iterated to convergence.

Note that for measurements at heights below about 10 m under moderately unstable conditions, it might be hard to distinguish between the logarithmic and modified profiles, to within the limits of experimental error. This is simply because ψ is of order zero for $Z << L$.

H. Penman method

It is very difficult to measure the surface latent heat flux (λE) using eddy correlation or profile methods, due to the problem of obtaining accurate and rapid response measurements of humidity. An alternative is Penman's method (1948), in which the latent heat flux can be deduced from single measurements of temperature, humidity, net radiation flux and windspeed at some height above the surface.

1. Resistance Notation

Before presenting the derivation of the Penman equation, a note on representing fluxes using resistance notation. A flux can be represented in a form analogous to *Ohm's Law*, *I=V/R*, i.e. current flows down a potential gradient at a rate determined by the resistance. Thus the momentum flux across a finite layer $\Delta z = (z_2 - z_1)$ can be expressed as

$$\tau = \frac{\rho(u_2 - u_1)}{r_a} \qquad (1)$$

where r_a is the *aerodynamic resistance*. If u_1 is taken to be the $u(z_0)=0$, then the resistance can be related to the log law

$$r_a = \frac{u(z)}{u_*^2} = \frac{\ln(z/z_0)}{ku_*} = \frac{\left[\ln(z/z_0)\right]^2}{k^2 u(z)}. \qquad (2).$$

Notice that the quantity in square brackets depends only on the nature of the surface: if that is known, then r_a is determined only by the single measurement of mean windspeed. For scalar fluxes, this leads to an expression predicting a scalar flux F_q in the form

$$F_q = u_*^2 \left(\frac{\phi_m}{\phi_q}\right) \frac{\Delta q}{\Delta u} \equiv \frac{\Delta q}{r_q}, \qquad (3)$$

in which $r_q = (\Delta u / u_*^2) \times (\phi_q / \phi_m)$ is the *aerodynamic resistance* of the layer (some authors use *conductance*, which is the reciprocal of r_q). In the case of momentum, $\phi_q = \phi_m$. For non-neutral conditions

$$r_a = \frac{\left[\ln(z/z_o) - \psi_m(z/L)\right]^2}{k^2 u}. \qquad (4).$$

Under *stable conditions*, the aerodynamic resistance for sensible heat transfer (r_h) is usually taken as equal to r_a. Under *unstable conditions*, the assumption $\phi_h = \phi_m^2$ means than we need to multiply r_a by ϕ_m to get r_h. In both cases, the aerodynamic resistance for water vapour and other scalar fluxes are generally assumed to be the same as that for sensible heat. Note that, in general, z should be replaced by (z-d) if the sampling height is not much greater than the zero-plane displacement d.

2. Penman-Monteith Equation

Firstly the sensible and latent heat fluxes should be written in resistance notation

$$H = -\rho c_p \frac{(T - T_0)}{r_h} \qquad (5)$$

$$\lambda E = -\frac{\lambda}{R_v T} \frac{(e - e_0)}{r_v} \qquad (6)$$

We can rewrite (6) in terms of r_h and the effective psychrometer constant, γ_*:

$$\lambda E = -\frac{\rho c_p (e - e_0)}{\gamma_* r_h}, \qquad (7)$$

where the "thermodynamic" psychrometer constant is given by $\gamma = c_p p / \lambda\varepsilon$ and has a value of 0.66 hPa K^{-1} near sea level, the ratio of gas constants is $\varepsilon = R_a / R_v$, and the effective psychrometer constant is $\gamma_* = \gamma r_v / r_h$.

The key to estimating the exchange of sensible and latent heat with the air above a surface is to establish the surface temperature, T_0. If this is not known, then it can be eliminated from the equations by assuming that the saturation vapour pressure is a linear function of temperature for small temperature differences. Additionally, it is assumed that the surface vapour pressure is at saturation, i.e.

$$e_0 = e_s(T_0) = e_s(T) - \Delta(T - T_0). \qquad (8)$$

By substitution of (8) into (7) we get λE in terms of T-T_0 which is not known. But by combining the new equation for λE , T-T_0 may be eliminated from (5) for H.

The final assumption is that of energy balance closure, i.e. that R_n-G=H+λE. By substitution of the new equation for H, we obtain the *Penman equation*

$$\lambda E = \frac{\Delta(R_n - G) + \rho c_p (e_s(T) - e)/r_H}{\Delta + \gamma_*}, \qquad (9)$$

giving latent heat flux as a function of variables measured at one height only. When considering evaporation from a vegetated surface, we call this equation the *Penman-Monteith equation* where the symbols have the same meaning, except that γ_* is an *apparent* psychrometer constant. Over crops the resistance to evaporation is larger than the resistance to heat transfer, due to *canopy resistance*. To allow for this, the effective psychrometer constant is usually assumed to be of the form

$$\gamma_* = \gamma\left(\frac{r_h + r_s}{r_h}\right) = \gamma\left(1 + \frac{r_s}{r_h}\right), \qquad (10)$$

where r_s is an effective surface resistance. The latter depends in a complicated way on soil moisture, type of vegetation, fractional cover, and time of year. It is usual to consider it as the result of a *crop* (or *canopy*) *resistance* (r_{sc}) and a *bare soil resistance* (r_{ss}) acting in parallel, so that

$$\frac{1}{r_s} = \frac{(1 - A)}{r_{sc}} + \frac{A}{r_{ss}}, \qquad (11)$$

where A is an effective fraction of bare soil area. Appropriate values of crop resistance are known for various types of vegetation. For 'moist' surface conditions during the day, r_{ss} is usually taken to be 100 s m^{-1}.

Finally, r_h can be approximated by the aerodynamic resistance r_a as it is more readily measureable. For observations close to the ground (e.g. below 3 m) the stability correction can be neglected, and r_a therefore estimated using (1).

The Penman approach does not require the 'special' equipment required for eddy correlation and profile methods, but has to make more assumptions about the transfer processes at the Earth's surface as well as the nature of the surface itself. Meaningful comparisons with H obtained by other methods can be made using data averaged over the order of 30 minutes (daytime conditions).

→ See Monteith and Unsworth (1990), pp 180-187, for useful background on the physical basis of the Penman-Monteith equation

→ Exercise: prove the equivalence of (6) and (7), remembering that the ideal gas law is $p = \rho R_a T$ **.**