# Comment on "An ensemble cumulus convection parameterisation with explicit cloud treatment" by T. M. Wagner and H.-F. Graf

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ABSTRACT

Wagner and Graf (2010) derive a population evolution equation for an ensemble of convective plumes, an analogue with the Lotka–Volterra equation, from the energy equations for convective plumes provided by Arakawa and Schubert (1974). Although their proposal is interesting, as the present note shows, there are some problems with their derivation.

## 1. Introduction

The Lotka-Volterra equations are a cornerstone of biological population dynamics and extensively-studied by biologists and mathematicians alike (e.g. Takeuchi 1996). For a system of interacting species $i$,

$$\frac{d}{dt}x_i = r_i x_i \left(1 - \sum_j a_{ij} x_j\right) \tag{1}$$

where the $r_i$ and $a_{ij}$ are constant coefficients and there are $x_i$ members of each species (each $x$ is assumed large and therefore it can be approximated by a real number).

It would be tempting to describe the evolution of an ensemble of convective plumes in analogous manner. Nober (2003); Nober and Graf (2005) proposed just such a system as a possible representation for convective clouds, by invoking the notion of a competition between the clouds in order to consume CAPE. More recently, Wagner and Graf (2010) constructed a derivation for a Lotka-Volterra system of interacting clouds, arriving at Eq. 10 below, which is an analogue to Eq. 1.

The purpose of the present note is to comment upon their derivation. Our notations closely follow those adopted by Wagner and Graf (2010, hereinafter WG10).

## 2. Derivation by WG10

In their derivation, WG10 begin with a pair of equations proposed by Arakawa and Schubert (1974, hereinafter AS74) describing the energy cycle of an ensemble of convective plumes. The pair consists of a budget equation for the cloud work function, $A_i$, for the $i$th type of convective plume (Eq. 142 of AS74):

$$\frac{d}{dt}A_i = F_i + \sum_j K_{ij} M_{bj}, \tag{2}$$

and an equation for evolution of the convective kinetic energy, $K_i$, for the $i$th type of convective plume (Eq. 132 of AS74):

$$\frac{d}{dt}K_i = A_i M_{bi} - \frac{K_i}{\tau_{\rm dis}}. \tag{3}$$

The system consists of $2N$ such equations if $N$ types of convective plumes are considered.

As we are concerned with a comparison to Eq. 1, it is worth remarking that Eq. 2 was developed by AS74 for the case of many clouds. If one considers a single cloud then of course its cloud–base mass flux will vary over the lifecycle of that cloud. However, within a large enough area of interest then there are many individual clouds belonging to each type. That means that an average over area effectively also imposes an average over the cloud lifecycle because the averaging extends over clouds at all the different stages of the lifecycle. Thus, all cloud variables in this article should be interpreted as lifecycle averages.

The cloud work function, $A_i$ defined by Eq. 133 of AS74 or Eq. 3 of WG10, measures the rate at which convective kinetic energy, $K_i$, is generated by the potential energy per unit of mass flux, $M_{bi}$, at the cloud base. The cloud work function, $A_i$, is itself in turn, forced by the large-scale at a rate $F_i$, and consumed at the rate $-K_{ij}$ by the $j$th type of convective plume per unit of its cloud–base mass flux, $M_{bj}$. We expect that the matrix $K_{ij}$ is *overall* negative, giving a net damping tendency (*cf.*, convective damping, Emanuel et al. (1994)), consistent with the physical reasoning just given above.

To produce a closed set of equations, WG10 further invoke a hypothesis for a relationship between convective kinetic energy and cloud-base mass-flux

$$K_i = \alpha_i M_{bi}^2, \tag{4}$$

as proposed by Pan and Randall (1998). This relationship assumes, in essence, that variations in the convective kinetic energy for the $i$th type are dominated by variations in vertical velocity rather than by any variations in the fractional area, $\sigma_i$, covered by clouds of that type. By substituting Eq. 4 into Eq. 3, we obtain

$$\frac{d}{dt} M_{bi} = \frac{1}{2\alpha_i} A_i - \frac{1}{2\tau_{\mathrm{dis}}} M_i. \tag{5}$$

Thus, Eqs. 2 and 5 together constitute a closed set of equations describing the evolution of an ensemble of convective plumes. Whereas Pan and Randall (1998) considered only the case with $N = 1$, the generalization to the $N$–plume case is straightforward. On the other hand, WG10 consider a further reduction of this system by introducing additional approximations.

A key approximation invoked by WG10 is a type of quasi–equilibrium. AS74 argued that by considering the large–scale forcing, $F_i$, of the cloud work function to be large compared to its actual rate of change $dA_i/dt$, then the cloud work function is maintained close to equilibrium with the large-scale forcing (see their Fig. 13). In that case, we may approximately set

$$F_i + \sum_j K_{ij} M_{bj} = 0 \tag{6}$$

(*cf.*, their Eq. 150). This approximation is known as convective quasi–equilibrium.

A second type of convective quasi–equilibrium can be introduced (*cf.*, Lord and Arakawa 1980) by applying a similar argument to the budget equation for the convective kinetic energy, Eq. 3. *i.e.*, assuming that the convective kinetic energy is rapidly damped as soon as it is generated by the cloud work function so that these two terms are in close balance:

$$A_i M_{bi} - \frac{K_i}{\tau_{\rm dis}} = 0. \tag{7}$$

Application of Eq. 4 to Eq. 7 leads to:

$$\frac{A_i}{M_i} \simeq \frac{\alpha_i}{\tau_{dis}} = \text{constant} \tag{8}$$

(*cf.*, Eq. 8 of WG10). Taking a time derivative of the above implies that

$$\frac{d}{dt}\left(\frac{A_i}{M_i}\right) \simeq 0$$

or

$$\frac{1}{A_i}\frac{dA_i}{dt} \simeq \frac{1}{M_i}\frac{dM_i}{dt} \tag{9}$$

(*cf.*, Eq. 9 of WG10).

By substituting Eq. 9 into Eq. 2, we obtain

$$\frac{dM_i}{dt} = \frac{F_i}{A_i} M_i + \sum_j \frac{K_{ij}}{A_i} M_i M_j \tag{10}$$

The above equation takes the same form as the Lotka–Volterra equation, Eq. 1, when the coefficients $F_i/A_i$ and $K_{ij}/A_i$ are treated as constants (or more precisely as slowly–varying large–scale variables).

WG10 make this claim stronger by regarding cloud number as proportional to cloud–base mass flux so that:

$$M_{bi} \propto \sigma_i \propto x_i \tag{11}$$

Here, $x_i$ is the number of convective plumes of the $i$th type, which must be proportional to the cloud fraction, $\sigma_i$, if the area occupied by an individual plume is constant for a given type $i$, as assumed by WG10. Using Eq. 11 in Eq. 10, the latter reduces to Eq. 1 with specific expressions for the coefficients, $r_i$ and $a_{ij}$.

## 3. Issues with the WG10 derivation

### *a. Timescale separation*

The first issue with WG10's derivation of population dynamics is in introducing convective quasi–equilibrium of the second kind, Eq. 7, whilst nonetheless maintaining a full prognostic description for Eq. 2: *i.e.*, , without also introducing convective quasi–equilibrium. For this procedure to be justified, the damping timescale, say $\tau_A$, of the cloud work function *implied* by Eq. 2 must be much longer than the timescale $\tau_{\text{dis}}$ describing damping of the convective kinetic energy in Eq. 3,

$$\tau_A \gg \tau_{\text{dis}}. \tag{12}$$

An expression for $\tau_A$ is deduced in the Appendix, based on a consideration of the eigenfrequencies of the system of Eqs. 2 and 5. In the absence of a robust physical theory to determine numerical values for $\tau_A$ and $\tau_{\text{dis}}$, we confine ourselves to the remark that the condition of Eq. 12 is certainly not obvious a priori. Note that Pan and Randall (1998) propose

$\tau_{\mathrm{dis}} \sim 10^3$ sec.

### b. *Self-consistency of the WG10 approximation*

The second issue is the assumption made by WG10 in Eq. 10 that $F_i/A_i$ and $K_{ij}/A_i$ can be treated as effectively constant. More precisely, they assume that $A_i$ is slowly–varying with

$$\left|\frac{1}{A_i}\frac{dA_i}{dt}\right| \ll \left|\frac{1}{M_i}\frac{dM_i}{dt}\right| \tag{13}$$

However, this is in clear contradiction with the constraint in Eq. 9 obtained from the assumed convective quasi–equilibrium of the second kind: *i.e.*, both the cloud work function and the convective kinetic energy evolve at the same rate in time. In other words, WG10 do not treat $A_i$ in a self-consistent manner.

We note in particular that when Eq. 8 is substituted into Eq. 10 then what originally looked like a nonlinear equation is revealed to be a linear equation:

$$\frac{dM_i}{dt} = \frac{\tau_{dis}}{\alpha_i}(F_i + \sum_j K_{ij}M_j) \tag{14}$$

We believe this to be the self-consistent description of the system under the approximation of convective quasi-equilibrium of the second kind rather than Eq. 10.

### c. *Interpretation of the WG10 system*

The final issue we wish to point out with the derivation found in WG10 concerns the interpretation of Eq. 10 as a system of plume population dynamics by invoking the relationship in Eq. 11. We interpret that relationship as an assumption that the evolution of cloud–base mass flux is controlled by that of the cloud fraction, and hence that of the plume number. However, this conflicts with the hypothesis of Eq. 4, which we interpeted as an assumption that the cloud fraction is independent of time.

Although this conflict is important for the interpretation given to Eq. 10, it is entirely separate from our points above concerning the validity of that equation. In fact, we note that some support for Eq. 11 can be found from cloud-resolving model experiments and scaling arguments. A summary is given in Sec. 2 of Plant and Craig (2008). Moreover, we also note that the hypothesis in Eq. 4 may not be in fact a critical part of the derivation in WG10. A more general hypothesis can be entertained within the same framework by replacing Eq. 4 with $K_i = c_i M_{bi}^p$, where now $c_i$ and $p$ are to be treated as constants. A hypothesis of that form for any $p > 1$, when combined with convective quasi-equilibrium of the second kind, still produces Eq. 10, albeit now with a factor of $p - 1$ on the left-hand side. Further discussions on an appropriate value for $p$ will be given elsewhere (Yano and Plant 2011).

# 4. Discussions

In spite of the issues that we have pointed out about the derivation of the plume population dynamics in WG10 from AS74's energy equations, these problems are probably inconsequential for their particular application.

In their study, they are only concerned with the evolution of a convective system in response to a time-evolving large-scale forcing. The sole purpose of the population dynamics, for this context, is to keep the convective system close to equilibrium. The derived population dynamics are indeed designed to perform this function, although the details of the transient behavior may not be quantitatively correct.

The equilibrium state could of course have been obtained directly from Eq. 6, by inverting the interaction matrix $K$. Integrating towards this state may be more convenient in practice. Moreover, the result of the integration procedure truncated at some suitably long time would not seem to be any worse than the ad-hoc procedures suggested by Lord and Arakawa (1980); Hack et al. (1984) for cases where a strict equilibrium does not hold and the matrix cannot be inverted.

With regards to the reply from the original authors (Wagner et al. 2011), readers must form their own judgement. We do point out, however, that they are completely silent about the main problem with their derivation which we identified in Sec. 3.b.

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# Appendix

In this Appendix, we derive eigenfrequencies for the full system of Eqs. 2 and 5 under a linear perturbation approach, and compare the result with that of the system of Eq. 14 for convective quasi-equilibrium of the second kind. For ease of demonstration, we consider the case of a single plume type, $N = 1$. In the following, therefore, we remove all subscripts $i$ labelling the cloud type from the notations of the main text.

Assuming an exponential decay rate, $\eta$, to a perturbation of the full system, we obtain two perturbation equations:

$$-\eta A' = KM' \tag{A1}$$

$$-\eta M' = \frac{1}{2\alpha}A' - \frac{1}{2\tau_{\rm dis}}M' \tag{A2}$$

where the dependent variables, $A'$ and $M'$, are perturbation quantities. Eliminating these variables, the eigenfrequencies are given by

$$\eta^2 - \frac{\eta}{2\tau_{\rm dis}} - \frac{K}{2\alpha} = 0 \tag{A3}$$

which is solved to give

$$\eta = \frac{1}{4\tau_{\rm dis}}\left[1 \pm \left(1 - \frac{8\tau_{\rm dis}}{\tau_A}\right)^{1/2}\right] \tag{A4}$$

where

$$\tau_A = \frac{\alpha}{|K|\tau_{\rm dis}} \tag{A5}$$

assuming $K < 0$.

The perturbation equation corresponding to Eq. 14 is

$$-\eta M' = \frac{\tau_{\rm dis} K}{\alpha} M', \tag{A6}$$

which immediately leads to a damping rate, $\eta = 1/\tau_A$.

Note that when $\tau_A \gg \tau_{\rm dis}$, the damping rates given by Eq. A4 are approximated by $\eta = 1/2\tau_{dis}$ and $1/\tau_A$ (in the latter case by retaining two terms in a Taylor expansion of the square root). The first damping rate corresponds to the fast damping process that ensures the convective quasi–equilibrium of the second kind, as given by Eq. 7. The second damping rate characterizes the slow evolution described by Eq. 14.

Thus, the damping rate from Eq. A6 agrees well with the slower rate from the full system of Eq. A4 only if the condition $\tau_A \gg \tau_{\rm dis}$ is satisfied. Otherwise, the reduced system under Eq. 14 would lead to substantial errors in characteristic frequencies.

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