Shallow Water Test Case on a Beta Plane with Dynamic Adaptivity

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1 Test Case Definition

This test case is to be undertaken using a variety of shallow water models which use different forms of dynamic adaptivity, such as nested refinement (often called adaptive mesh refinement, AMR or non-conforming refinement), unstructured refinement (using arbitrarily structured grids of triangles or polygons) or r-adaptivity (mesh movement or mesh redistribution without changing the topology). Since not all of the participating models represent a sphere, a barotropically unstable jet on a β plane is used. The jet is defined to be similar to the Galewsky et al. [2004] jet but on a plane rather than on a sphere.

The models should all solve the non-linear shallow water equations on a β plane. They are given here in advective form but this can be model dependent:

$$
\begin{aligned}
\frac{\partial u}{\partial t} + \mathbf{u}\cdot\nabla u &= fv - g\frac{\partial h}{\partial x} \\
\frac{\partial v}{\partial t} + \mathbf{u}\cdot\nabla v &= -fu - g\frac{\partial h}{\partial y} \\
\frac{\partial h}{\partial t} + \mathbf{u}\cdot\nabla h &+ h\nabla\cdot\mathbf{u} = 0
\end{aligned}
$$

where u and v and the velocities in the x and y directions, $\mathbf{u} = \begin{pmatrix} u \\ v \end{pmatrix}$, h is the shallow water height and $f = \beta y$ is the Coriolis and g is the acceleration due to gravity.

1.1 Analytic solution: a geostrophically balanced, compact jet

An analytic solution of the shallow water equations on a β plane consisting of a compact, zonal jet with zero first and second derivatives at the edges is given by:

$$
\begin{aligned}
u(\hat{y}) &= \begin{cases} u_0\left(1-3\hat{y}^2+3\hat{y}^4-\hat{y}^6\right) & \text{for } -1\le\hat{y}\le 1 \\ 0 & \text{otherwise} \end{cases} \\
v &= 0 \\
h\left(\hat{y}\right) &= \begin{cases} h_0 & \text{for } \hat{y}\le -1 \\ h_0-\frac{w\beta u_0}{g}\left\{y_c\left(\frac{16}{35}+\hat{y}-\hat{y}^3+\frac{3}{5}\hat{y}^5-\frac{1}{7}\hat{y}^7\right)\right. & \text{for } -1\le\hat{y}\le 1 \\ \qquad\qquad \left. +w\left(-\frac{1}{8}+\frac{1}{2}\hat{y}^2-\frac{3}{4}\hat{y}^4+\frac{1}{2}\hat{y}^6-\frac{1}{8}\hat{y}^8\right)\right\} & \\ h_0-\frac{32}{35}\frac{w\beta u_0 y_c}{g} & \text{for } \hat{y}\ge 1 \end{cases}
\end{aligned}
$$

where $\hat{y}=\frac{y-y_c}{w}$, y_c is the y position of the jet centre, w is the jet half width, u_0 is the maximum jet velocity and h_0 is the minimum height. This is the same analytic solution used for the test case with static refinement.

1.2 Initial Perturbation

In order to trigger barotropic instability in a reproducible way, an initial perturbation is added to the height in a similar manner to Galewsky et al. [2004]:

$$
h\left(x,y\right)\to h\left(\hat{y}\right)+\hat{h}\cos^2\left(\frac{\pi}{2}\hat{y}\right)e^{-\left(\frac{x-x_c}{w_x}\right)^2}e^{-\left(\frac{y-y_c}{w_y}\right)^2} \tag{1}
$$

where $\hat{h}$ is the maximum perturbation height, (x_c,y_c) is the centre of the perturbation (being co-located with the jet centre) and w_x and w_y are the scaling widths of the perturbation in the x and y directions. The values of all parameters are given in section 1.4.

1.3 Geometry

The domain extends a distance $8a$ in the x direction and $2a$ in the y direction with $y=0$ representing the equator and thus has $f=0$. The boundaries at $y=0,\ 2a$ are no normal flow ($v=0$) and the boundaries at $x=0,\ 8a$ are periodic. This test case can be run on an initially completely aligned, orthogonal grid. However in order to mimic the distortions in a mesh of the sphere that occur, for example, in a cubed sphere, a mesh should include grid-lines which change direction by at least 60^o such as in figure 1. The vertices and vertex connectivity of this mesh is given at `http://www.met.reading.ac.uk/~sws02hs/inProgress/NewtonSWtestCases/longMesh.obj` (with vertices numbered from 1). The mesh used for a numerical simulation can be refined from this mesh. If hexagonal or triangular meshes are used they should also reflect the need to mesh the sphere so there should be at least 5 pentagons in a hexagonal mesh or five triangles meeting at a point for triangular meshes.

1.4 Parameter Values

All of the parameter values are defined in table 1. Jun-ichi Yano (personal communication, 2012) has estimated that, for low Froude number, this jet will be barotropically unstable if $\frac{w^2\beta}{u_0}<\frac{4}{5}$. The parameter values are chosen so that we are close to but below this limit. Without the initial perturbation, these parameter values give the profiles of height, velocity and relative vorticity (ξ) shown in figure 2.

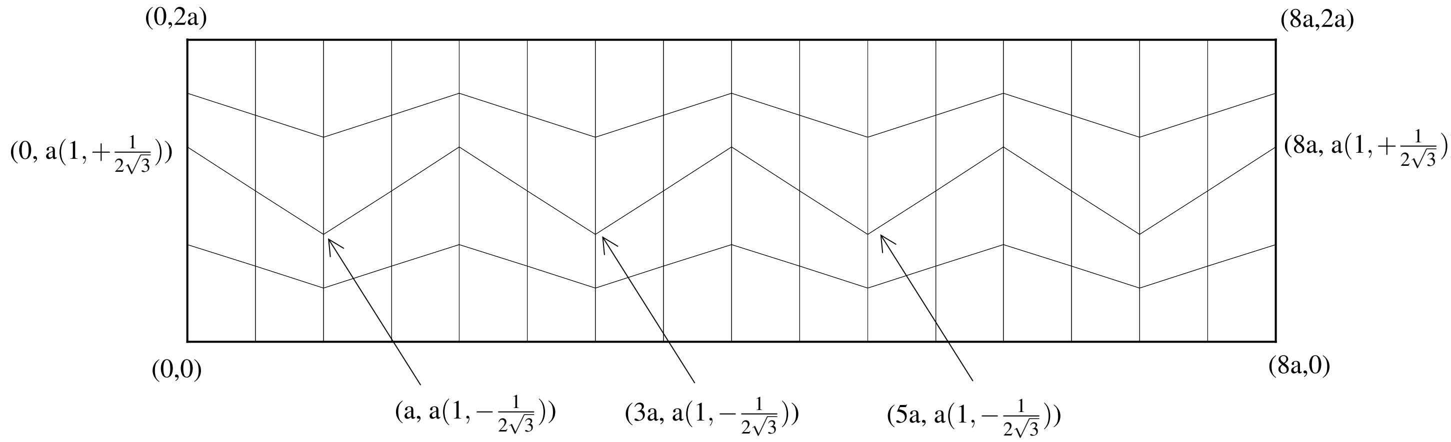

Figure 1: The block structure of a mesh on a plane with angles similar to a cubed sphere.

Gravitational acceleration	g	10	ms^{-2}
Change of Coriolis with y	β	2×10^{-11}	$\mathrm{s}^{-1}\mathrm{m}^{-1}$
Maximum wind speed	u_0	80	ms^{-1}
Geometry parameter	a	6×10^{6}	m
Jet width parameter	w	$\frac{a}{4}$	m
Jet centre	y_c	a	m
Minimum shallow water height	h_0	3000	m
Perturbation maximum height	$\hat{h}$	120	m
Perturbation centre	(x_c, y_c)	$(4a, a)$	(m,m)
Perturbation scaling widths	(w_x, w_y)	$(1.5\times10^{6}, 2\times10^{6})$	(m,m)
Total run length	T	24	days
Domain limits		$(0\rightarrow 8a,\ 0\rightarrow 2a)$	m

Table 1: Parameter values for the mesh adaptivity, shallow water test

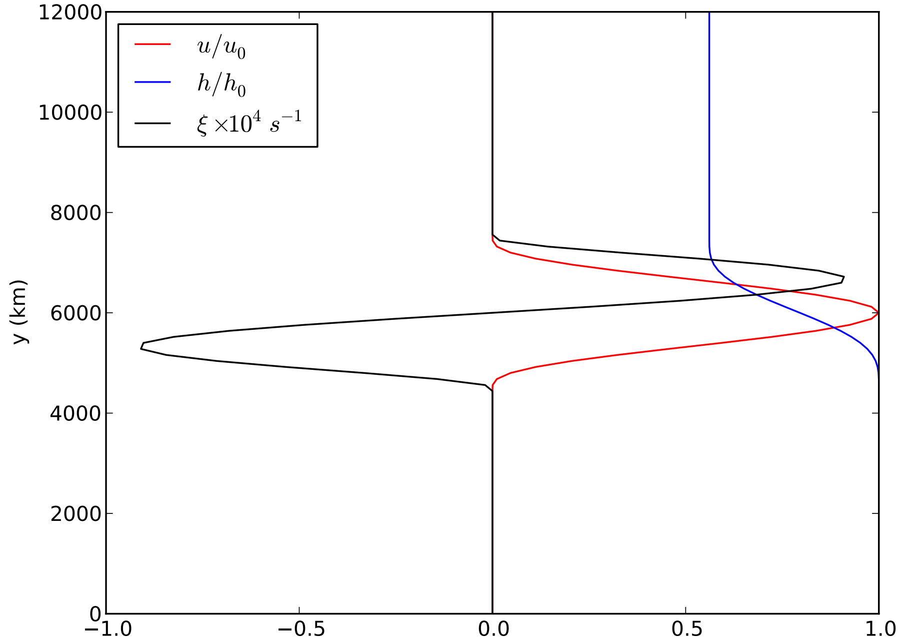

Figure 2: Values of velocity, height and relative vorticity for the initial, unperturbed jet

2 Results

The relative vorticity ξ in the entire domain is shown in figures 3 and 4 for days 1 to 20 after initialisation. The simulation in figure 3 uses a regular, orthogonal and uniform Cartesian grid of 400×100 finite volumes using a C-grid staggered version of OpenFOAM with quadratic differencing. It is therefore not a challenge to reproduce the unperturbed stretch of the jet out to day 7 or 8 using this model. The simulation in figure 4 uses a cubed-sphere like grid of 400×100 finite volumes based on the structure in figure 1 and with velocity and height co-located with Rhie and Chow [1983] regularization and quadratic differencing. With the non-orthogonal cells, the non-alignment of the initial jet with the mesh and the abrupt changes of direction of the mesh, the barotropic instability is released 3 days earlier than it should be. This is thus a significant challenge for any non lat-lon model (or a challenge for a lat-lon model if the jet were rotated with respect to the lat-lon grid).

The evolution of the release of instability can be summarised by calculating the root mean square of the gradient of the vorticity in the x direction ($\frac{1}{A}\sqrt{\sum_i \left(A_i \frac{\partial \xi}{\partial x}_i\right)^2}$ where A is the total area, the sum is over all the areas on which $\frac{\partial \xi}{\partial x}$ is calculated and A_i is each area). This diagnostic is shown in figure 5 for the simulations using both grids as a function of time. The total number of degrees of freedom divided by three per time step should also be reported (as in figure 5). The total number of degrees of freedom includes degrees of freedom for height and velocity. Hence it is divided by three to give (approximately) the number of computational points. This value should also be summed for each time step to give the total number of degrees of freedom used over the whole simulation. For a fixed grid this is simply the degrees of freedom multiplied by the number of time steps. The simulations presented use a time step of 300s for 24 days and 400×100 cells so the integrated degrees of freedom is $3 \times \frac{24 \times 86400}{300} \times 400 \times 100 =$

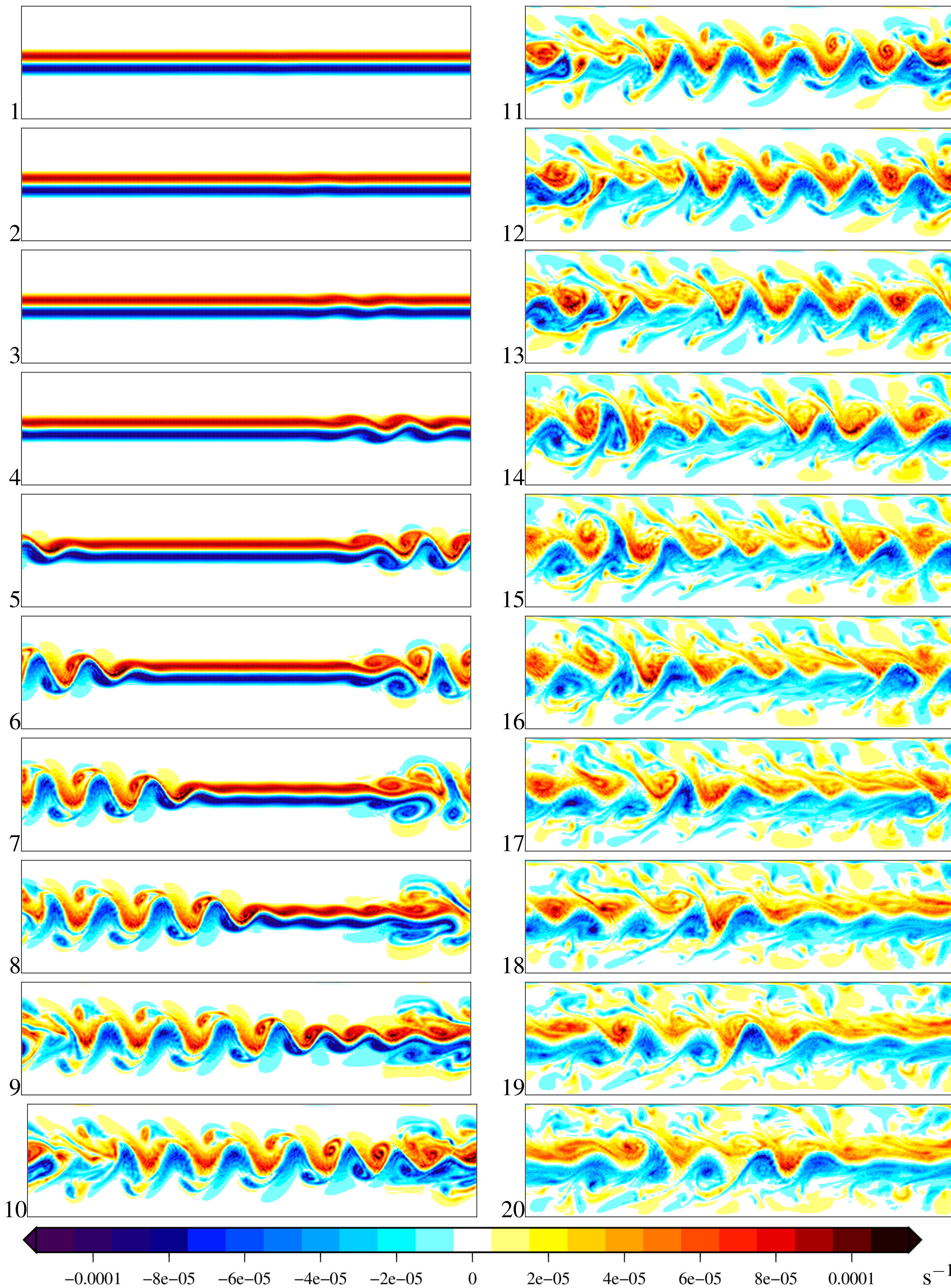

Figure 3: Relative Vorticity of the barotropically unstable jet calculated using an orthogonal mesh of 400×100 finite volumes using a C-grid staggered version of OpenFOAM with quadratic differencing from day 1 to 20 after initialisation.

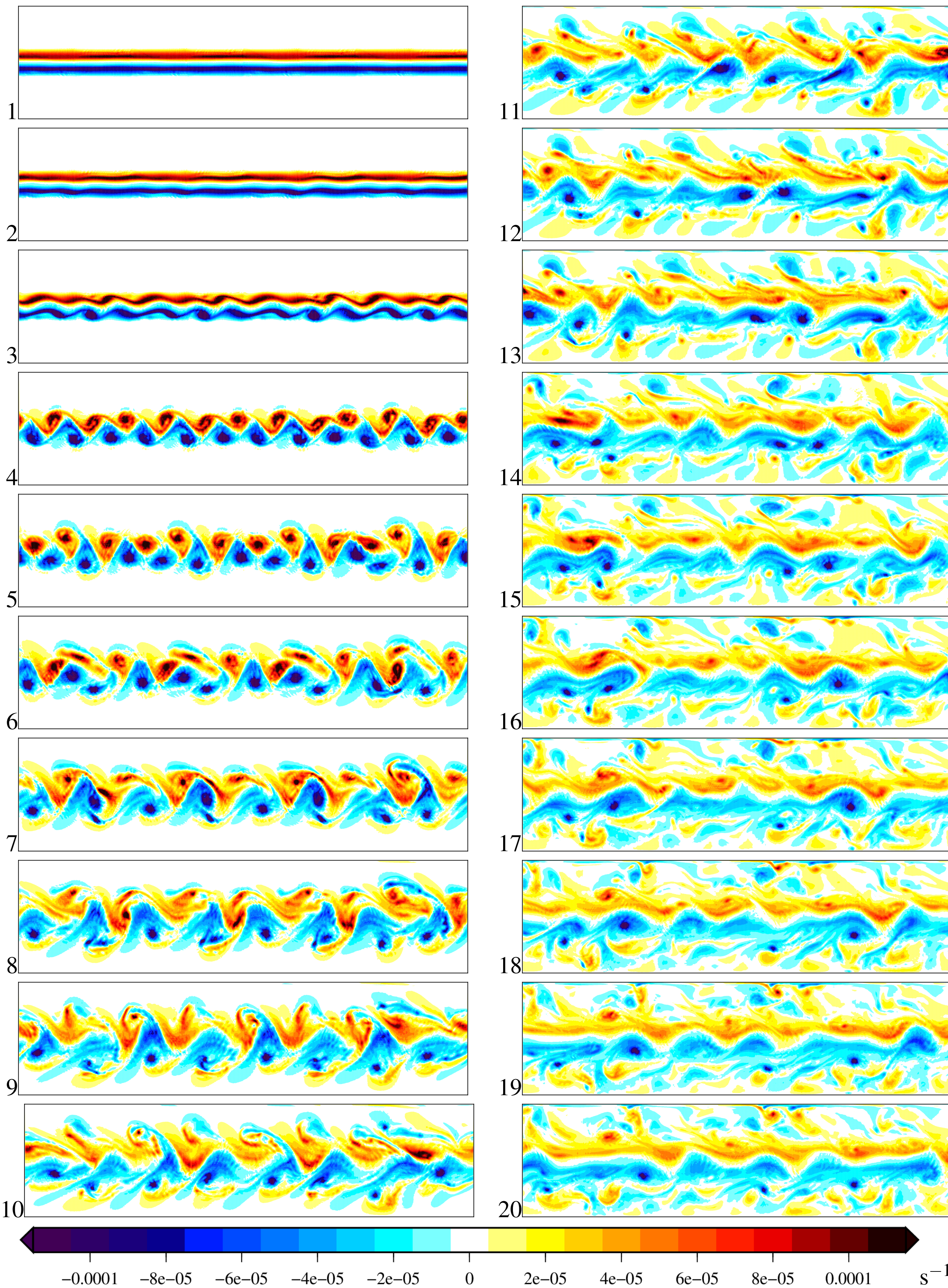

Figure 4: Relative Vorticity of the barotropically unstable jet calculated using a non-orthogonal, cubed-sphere like mesh of 400×100 finite volumes using a co-located version of OpenFOAM with quadratic differencing from day 1 to 20 after initialisation.

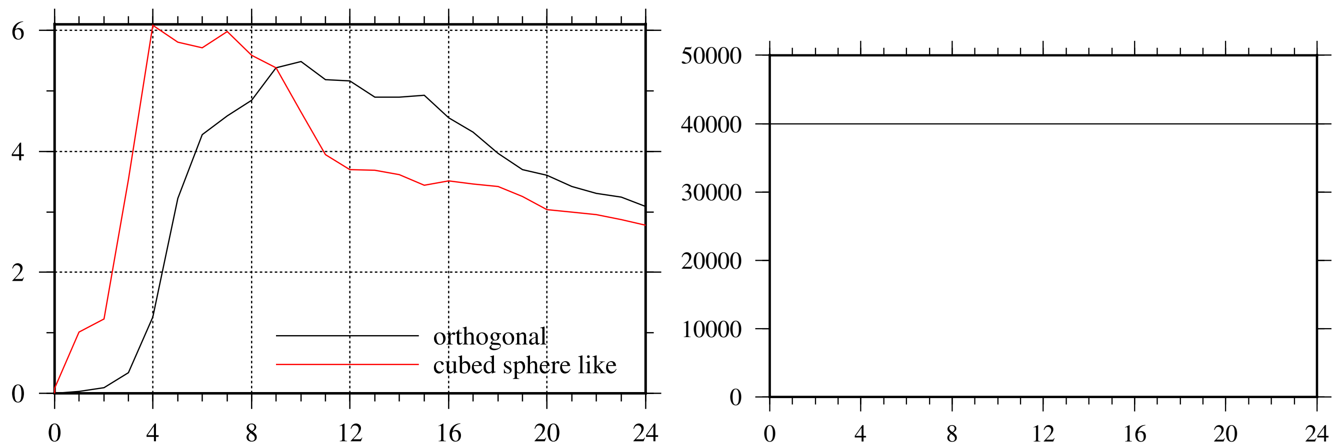

Figure 5: Left: root mean square gradient of the vorticity in the x direction as a function of days for the simulations using both grids. The vorticity gradient is multiplied by 10^{11} and has units $m^{-1}s^{-1}$. Right: a third of the total number of (height plus velocity) degrees of freedom per time step (ie the number of computational points) as a function of day. This is constant since these simulations do not use dynamic adaptivity. The integrated number of degrees of freedom (ie degrees of freedom $\times$ number of time steps) over the 24 days is $3 \times 2.7648 \times 10^8$.

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3 Summary of the Numerical Method

The salient points of the numerical method, the remeshing strategy, mesh to mesh interpolation and the refinement criterion should be listed with references to more detailed descriptions. The summary should include the following:

- Discretisation type (eg finite element, discontinuous Galerkin, finite volume, spectral etc)
- Prognostic variables used and their placement on the mesh
- Number of degrees of freedom per element or stencil sizes for the operators
- Order of accuracy of the approximations for all of the spatial operators and special techniques for each operator
- Theoretical mimetic properties
- Mimetic properties of the mesh to mesh interpolation (ie what is conserved)
- A brief description of the mesh generation technique used for re-meshing
- Details of any special coupling between mesh regions of different resoluiton
- Details of the refinement criterion and the re-meshing frequency.
- Any limiters used
- Time-stepping schemes for each of the terms
- Time steps used for each term of the equations
- Maximum theoretical time-steps

- The size, number and condition number of any matrix solutions (eg mass matrices or linearised Helmholtz equations)
- The linear equation solvers and pre-conditioners
- Number of calls to the solvers per time step
- Typical number of iterations per solver call

References

J. Galewsky, R. Scott, and L. Polvani. An initial-value problem for testing numerical models of the global shallow-water equations. *Tellus*, 56A(5):429–440, 2004.

C. Rhie and W. Chow. Numerical study of the turbulent-flow past an airfoil with trailing edge separation. *AIAA Journal*, 21(11):1525–1532, 1983.