Curl-free pressure gradients

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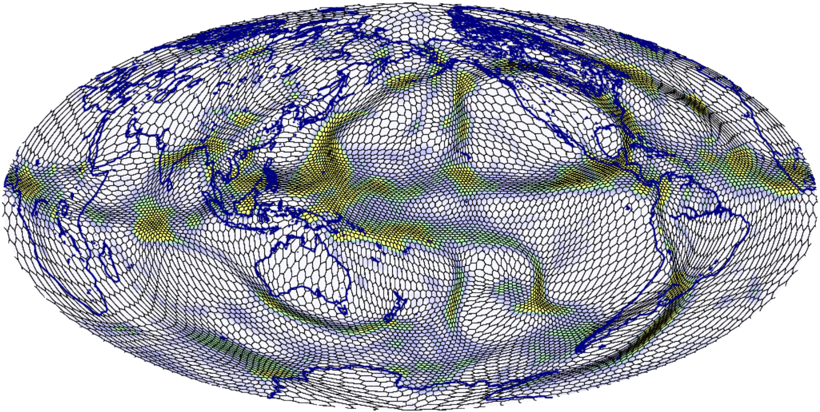

Based on work published in

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Why Curl-Free Pressure Gradients

- Take the curl of the vector invariant momentum equation to get the vorticity equation:

$$\frac{\partial \nabla \times \mathbf{u}}{\partial t} + \nabla \times (\mathbf{u} \cdot \nabla \mathbf{u}) + \nabla \times (2\Omega \times \mathbf{u}) = \nabla \times \mathbf{g} - \nabla \times \frac{1}{\rho} \nabla p$$

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- A discrete mimic of this phenomenon is a property of C-grid models that solve the vector invariant form of the momentum equation.
  - Normal component of pressure gradient calculated at velocity points using simple finite difference
  - Vorcitity at vertexes, calculated from Stoke's circulation theorem from normal velocity

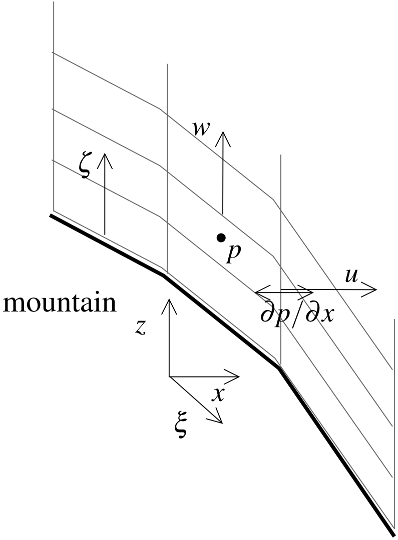
Curl-Free Pressure Gradients on the Cubed-Sphere

This also works on non-orthogonal grids such as the cubed-sphere

- Use co-variant rather than contra-variant components of velocity
- Calculate pressure gradient in the same direction
- Hodge operator, H, to transform from co-variant to contravariant velocities (Contra-variant velocities needed to solve the continuity equation conservatively)

Curl-Free Pressure Gradients over Orography

But when we use terrain-following coordinates, this property is usually lost



Momentum equation becomes:

$$
\begin{aligned}
\frac{\partial \mathbf{u}}{\partial t} + \cdots &= -\frac{1}{\rho}\nabla p \\
&= -\frac{1}{\rho} J \nabla p
\end{aligned}
$$

where J is the Jacobian of the transform $(x,z) \to (\xi,\zeta)$. Numerically it is easy to achieve $\nabla \times \nabla p = 0$ but not

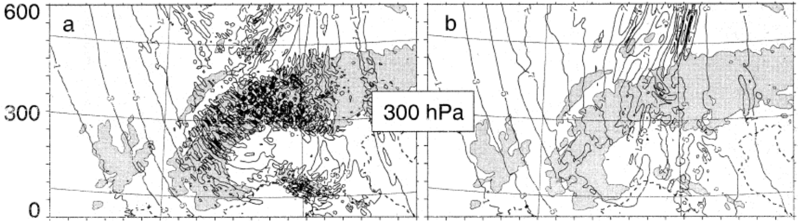
$$\nabla \times (J\nabla p) = 0$$

.

Vorticity Generation Using Terrain-Following Coordiantes

Potential vorticity over Alps in two models from Hoinka and Zängl [2004]

$\nabla \times \nabla p \neq 0$ $\quad\quad$ $\nabla \times \nabla p$ much smaller



Numerical curl in the pressure gradient where the mesh is non-orthogonal over orography leads to spurious vorticity generation near the tropopause

Numerical Solution of the Euler Equations

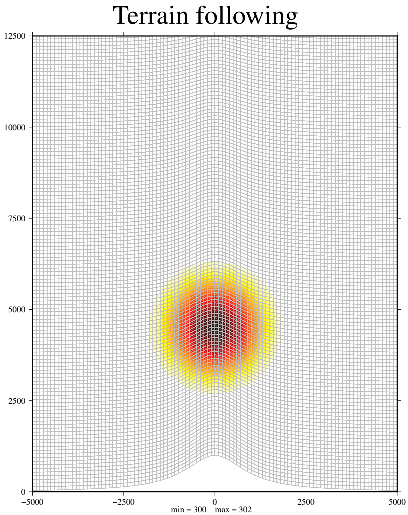
$$\begin{aligned}
\frac{\partial \rho \mathbf{u}}{\partial t} + \nabla \cdot \rho \mathbf{u}\mathbf{u} &= \rho \mathbf{g} - c_p \rho \theta \nabla \Pi \\
\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \mathbf{u} &= 0 \\
\frac{\partial \rho \theta}{\partial t} + \nabla \cdot \rho \mathbf{u} \theta &= 0 \\
\Pi^{\frac{1-\kappa}{\kappa}} &= R\rho\theta / p_0
\end{aligned}$$

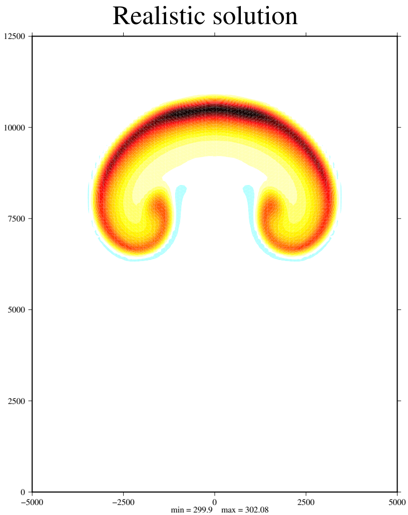
where $\Pi = (p/p_0)^\kappa$, $\theta = T(p_0/p)^\kappa$.

- Finite-volume C-grid
- Numerical solution for ρu, ρ and $\rho\theta$
- Implicit treatment of gravity and acoustic waves
- No reference profile
- Lorenz staggering
- Multi-dimensional cubic-upwind fit advection

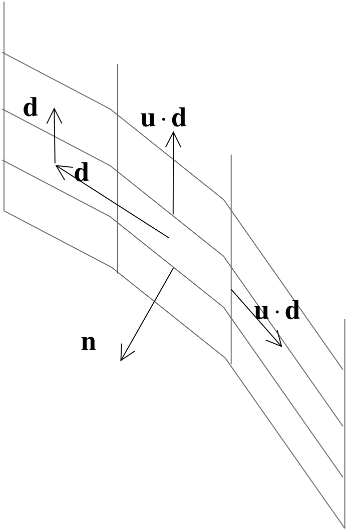
Warm Bubble Rising Over Orography [Good et al., 2013]

Final maximum Courant number, 0.9 ($\Delta t = 5s$)





Solution: use covariant components of velocity [Weller and Shahrokhi, 2014]



Prognostic variables: $u_d = \mathbf{u} \cdot \hat{\mathbf{d}}$

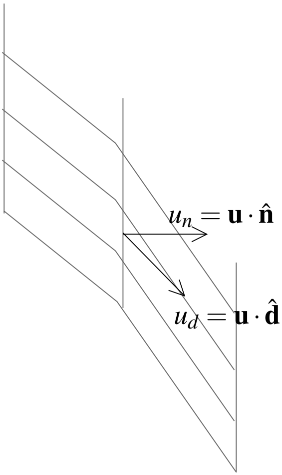
where $\mathbf{d}_e = \mathbf{x}_i - \mathbf{x}_j$

$\rightarrow$ curl free pressure gradients

$\rightarrow$ no spurious generation of vorticity

THIS IS HOW IT IS DONE IN THE HORIZONTAL. DO THE SAME THING IN THE VERTICAL!

Solution: use covariant components of velocity [Weller and Shahrokhi, 2014]



- Still need $u_n = \mathbf{u} \cdot \hat{\mathbf{n}}$ in continuity equation
- Hodge operator: $u_n = H u_d$
- For energy conservation, H must be symmetric and positive definite [as in the horizontal, see Thuburn et al., 2014]
- First reconstruct full velocity at cells, c from $u_{d'}$ at the surrounding faces, f':

$$\mathbf{u}_c = T_i^{-1} \sum_{f'} \mathbf{d}_{f'} A_{f'} u_{d'}$$

$$\text{where } \; T = \sum_{f'} \hat{\mathbf{d}}_{f'} \hat{\mathbf{d}}_{f'}^T A_{f'}$$

(Least squares fit)

- Then for face f between cells c and c':

$$u_n = \frac{1}{2} \left(\mathbf{u}_c + \mathbf{u}_{c'} \right) \cdot \hat{\mathbf{n}}$$

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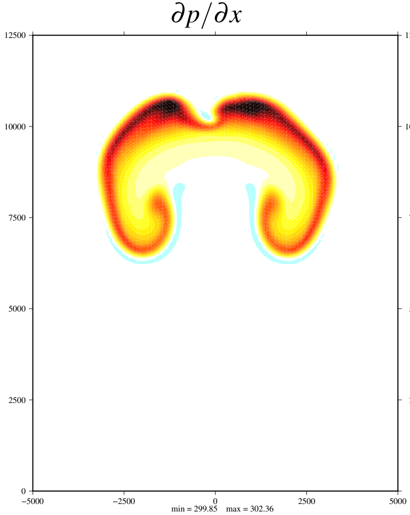
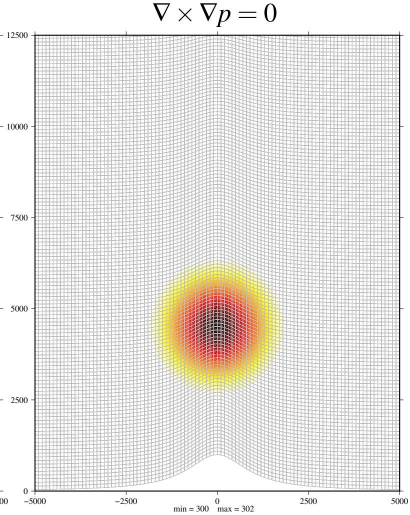
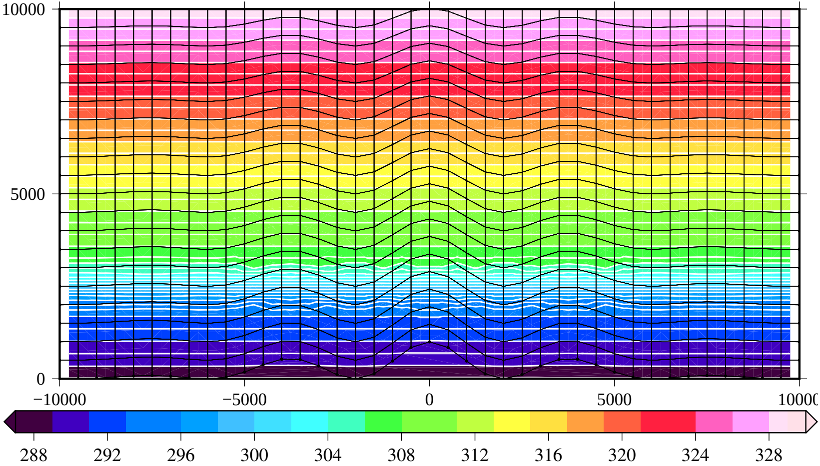




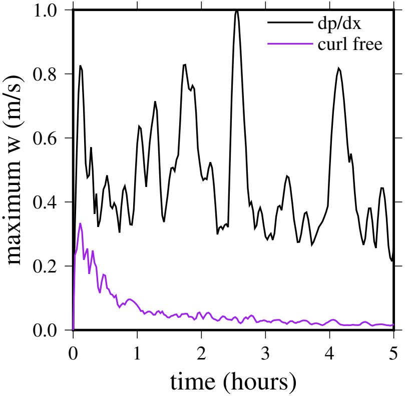
Stably Stratified Flow Over Orography [Klemp, 2011]

Atmosphere initially at rest should remain at rest. Potential temperature, θ:

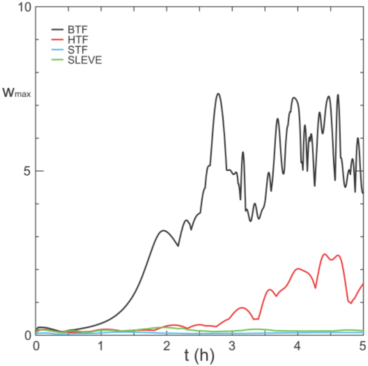


Spurious vertical velocity as a function of time

My results ($\Delta t = 100$s)



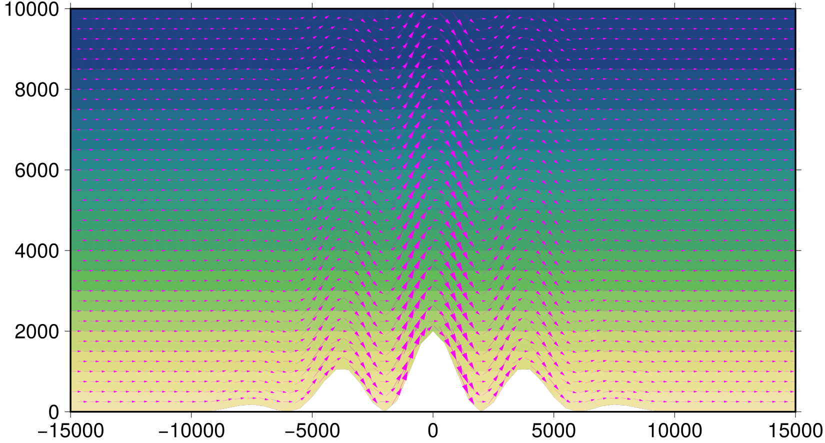
From Klemp [2011]



A Test Case to Challenge Cut Cells

See James Shaw's poster and Shaw and Weller, (MWR, 2016)

- Terrain following wind
- Advection of a stratified tracer: initial conditions
- Analytic solution available



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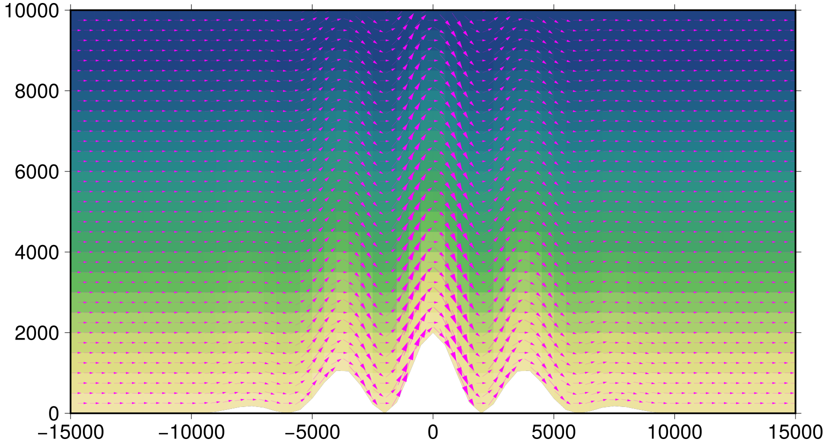
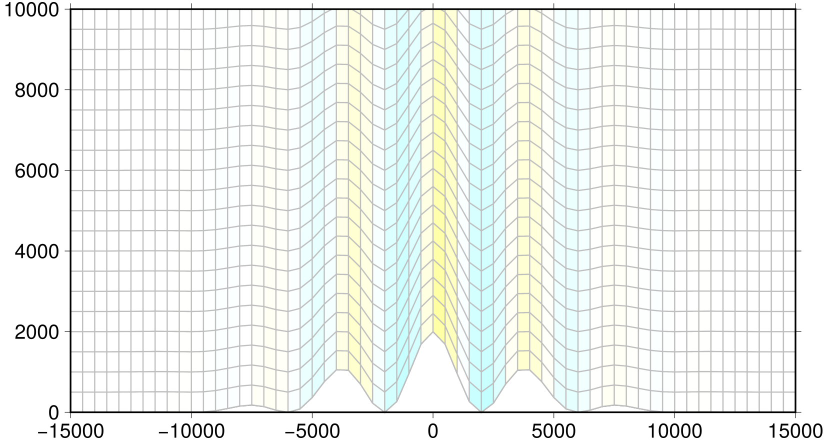
- Steady state solution



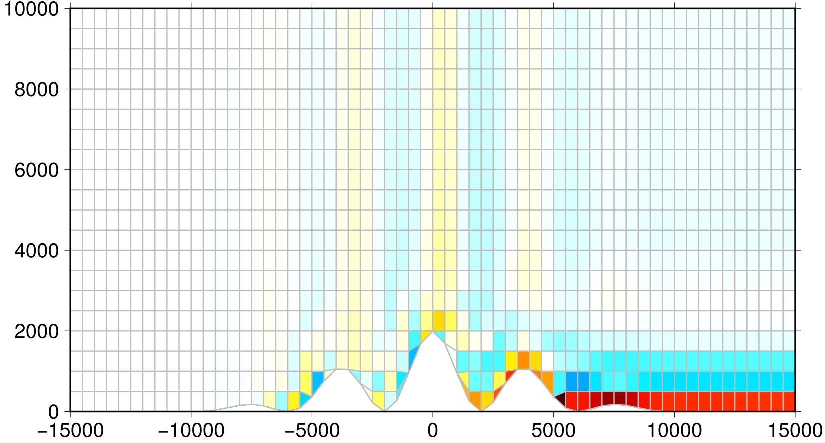
Errors on a Terrain Following Grid

- Flow and grid are aligned
- $\rightarrow$ very low errors



A new type of Cut Cell Grid

- Very simple grid construction algorithm
- Avoids cells which are small in the direction of flow



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- New advection test case with analytic solution to challenge cut cell grids

The End