Adaptive Modelling of the Atmosphere using Optimal Transport

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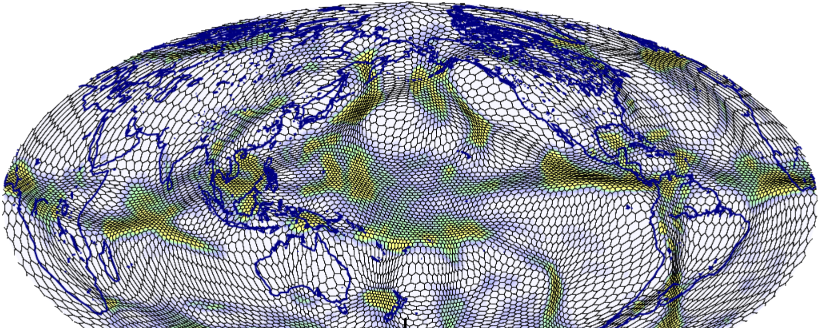
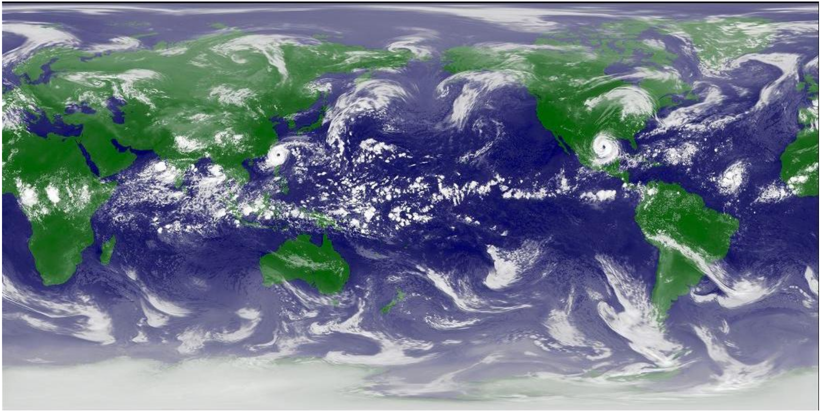
University of Reading
METEOROLOGY

21 March 2018



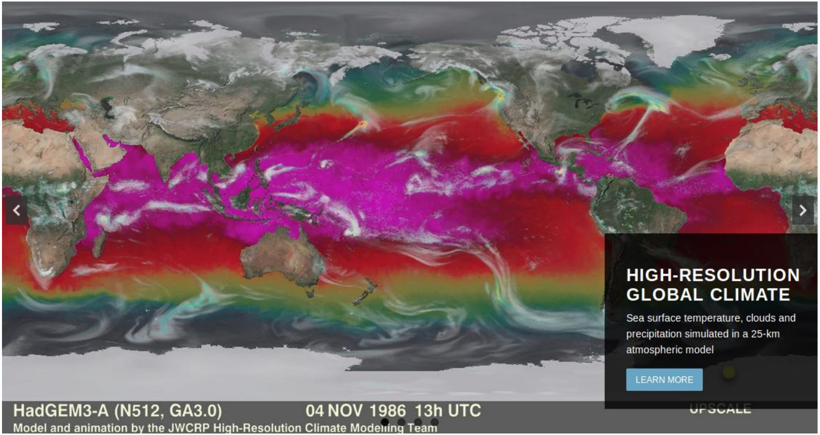
NICAM at 7km resolution

http://nicam.jp/hiki/?About+NICAM



HadGEM at 25km resolution

https://hrcm.ceda.ac.uk/



Why Mesh Adaptivity?

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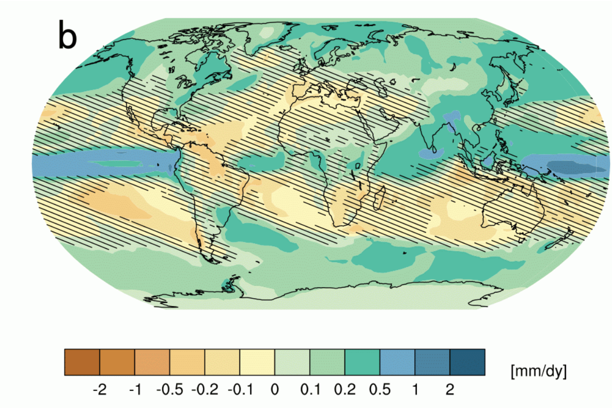
- Regional predictions of climate change



Precipitation changes under RCP 4.5 from CMIP5 multi-model mean. Hatching where the multi-model mean change is less than the natural variability. From Xie et al. [2015]

r-Adaptivity - **R**edistribution

- keep mesh topology fixed
- deform the mesh based on a monitor function

r-Adaptivity - Redistribution

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Fiedler and Trapp [1993] A thermal using the CGDA model. Mesh calculated using iterative relaxation on a coarser mesh

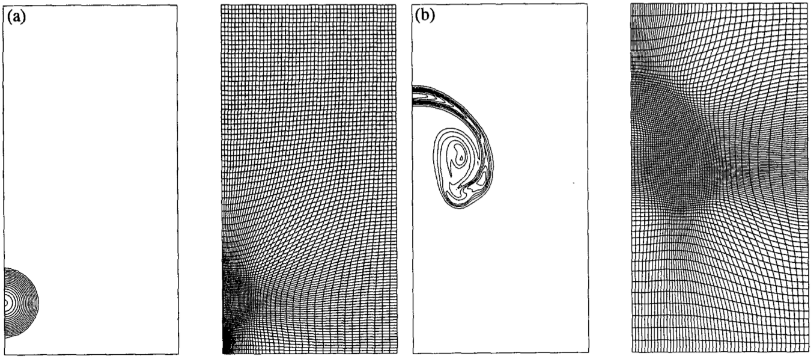
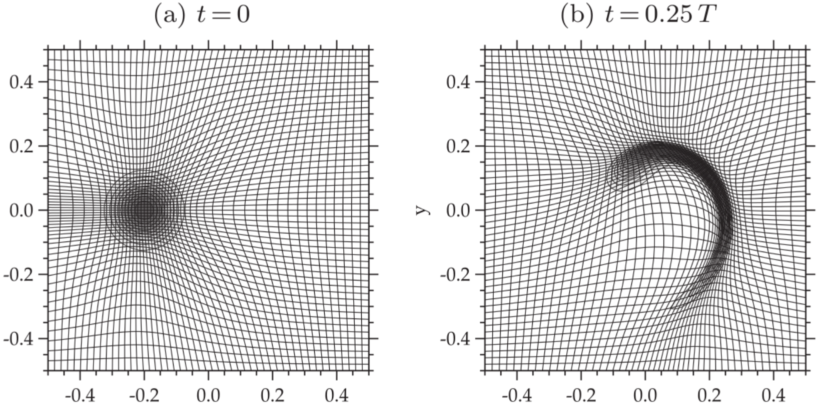


FIG. 2. Buoyancy b and u nodes of the full grid at (a) $t = 0$ and (b) $t = 4$ in the two-dimensional 49 × 97 CDGA model. The contour interval is 0.06. The maximum buoyancy values are 1.00 and 0.554 at $t = 0$ and $t = 4$, respectively. All the major extrema are maxima.

r-Adaptivity - **R**edistribution

Kühnlein et al. [2012] - MPDATA scalar advection, mesh calculated using a parabolic moving mesh PDE



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- First, how do we define the deforming mesh?

Optimally Transported Meshes in Euclidean Geometry

- How to create a mesh which is equidistributed with respect to a monitor function. ie

$$A_x m(\mathbf{x}) = \text{const}$$

for each cell with area A_x for mesh monitor function $m(\mathbf{x})$.

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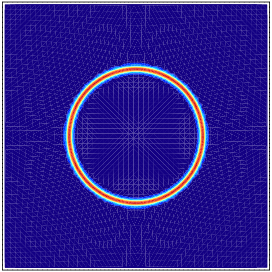
- This works fine in Euclidean geometry ...

Numerical Solution of the Monge-Ampère Equation on a Plane (test case from Budd et al. [2015])

Monitor Function, m



Initial Mesh

Solution of
$|I + H(\phi)|m(\mathbf{x}) = \text{const}$

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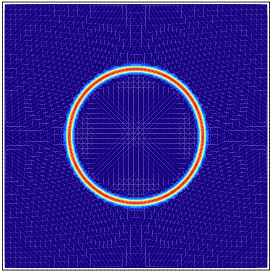	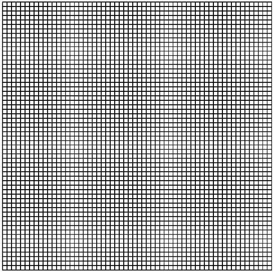	

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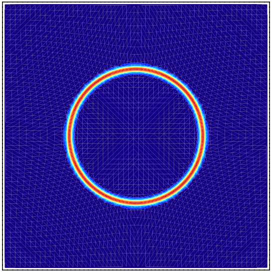

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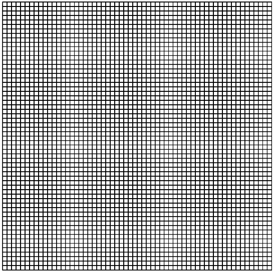

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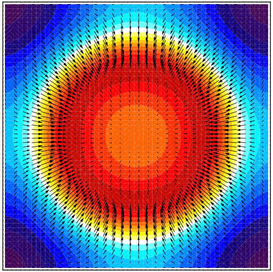
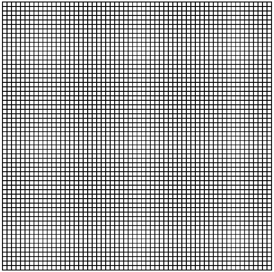


Finite volume discretisation in space
Fixed point (Poincaré) outer iterations

Initial Mesh + $\nabla\phi$ gives an equidistributed mesh

Initial Mesh, $\boldsymbol{\xi}$ $+\nabla\phi$ $=\mathbf{x}$



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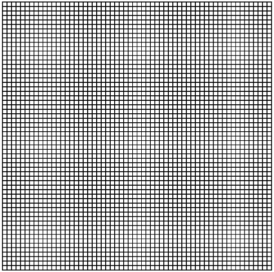

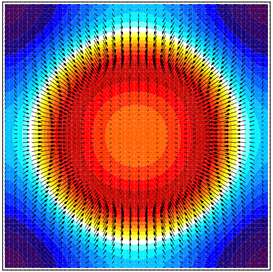

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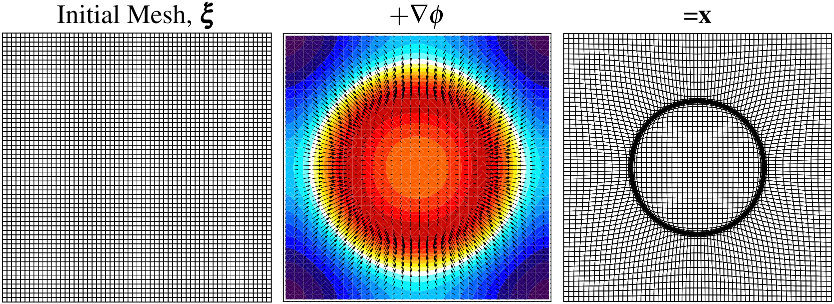

But on the Surface of a Sphere

- The gradient of a potential does not map to point on the sphere
- $\det(\nabla\nabla\phi)$ does not give change in area
- $\therefore$ cannot formulate a Monge-Ampère equation for the mesh potential.

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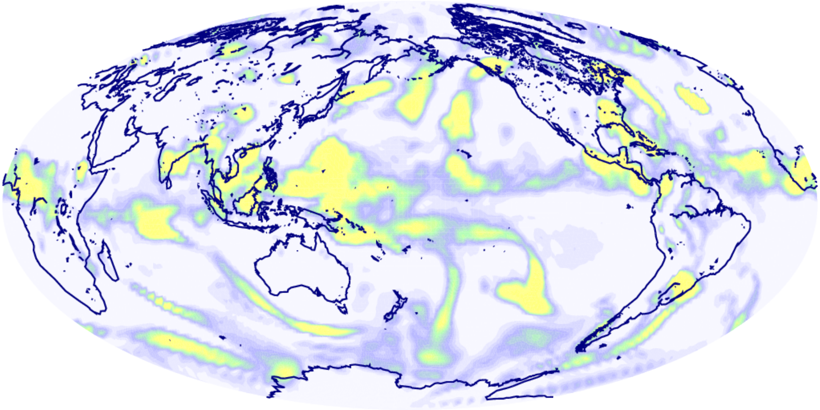
- Create Poincaré iterations with non-linear terms treated explicitly:

$$1 + \nabla^2\phi^{n+1} = 1 + \nabla^2\phi^n - \frac{A_x}{A_\xi} + \frac{\text{const}^n}{m\left(\mathbf{x}^n\right)}$$

Precipitation as a Monitor Function

$$m = \frac{p + p_{\min}}{p_{\max} + p_{min}} \text{ where } p_{\min} = 10^{-5}\text{kgm}^{-2}\text{s}^{-1},\ p_{\max} = 8.73 \times 10^{-4}\text{kgm}^{-2}\text{s}^{-1}$$



Using daily average precipitation rate, 1-12 Oct 2012, from the NOAA-CIRES 20th Century Reanalysis version 2 (Compo et al, 2011, `http://www.esrl.noaa.gov/psd/data/gridded/data.20thC_ReanV2.html`)

See Weller, Browne, Budd, and Cullen [2016]

Finite Volume Solution of PDEs on Moving Meshes

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Combine Reynolds transport theorem:

$$\frac{d}{dt}\int_{\mathscr{V}(t)} \rho \, dV = \int_{\mathscr{V}(t)} \frac{\partial \rho}{\partial t} \, dV + \int_S \rho \, \mathbf{v} \cdot \mathbf{dS}$$

with the continuity equation integrated over a volume:

$$\int_{\mathscr{V}(t)} \left\{ \frac{\partial \rho}{\partial t} + \nabla \cdot (\mathbf{u}\rho) \right\} = 0.$$

Discretise in time and apply Gauss's divergence theorem:

$$\frac{\mathscr{V}^{n+1}\rho^{n+1} - \mathscr{V}^n \rho^n}{\Delta t} = -\int_S \rho \, (\mathbf{u} - \mathbf{v}) \cdot \mathbf{dS}.$$

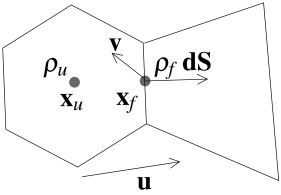
Definitions:

ρ	density
$\mathbf{u}$	fluid velocity
$\mathscr{V}(t)$	cell volume
$S(t)$	cell surface
$\mathbf{dS}$	outward normal
$\mathbf{v}(t)$	mesh velocity
$(\mathbf{u}-\mathbf{v}) \cdot \mathbf{dS}$	Volume flux

Discretisation

Definitions:

n	Time-step number
Δt	Time-step
ρ'	Temporary value of ρ^{n+1}
ϕ_r	Relative face flux
	$= (\mathbf{u}-\mathbf{v})\cdot\mathbf{dS}$



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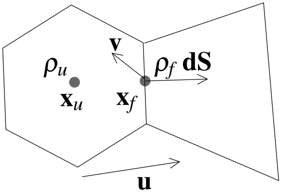

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\mathscr{V}^{n+1}\rho^{n+1} &= \mathscr{V}^{n}\rho^{n} - \frac{\Delta t}{2}\left(\sum_{\text{faces}} \rho_f^n \phi_r + \sum_{\text{faces}} \rho_f' \phi_r\right)
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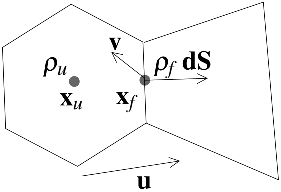

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- Discretise in space using OpenFOAM's finite volume linear upwind advection scheme.

$$\rho_f = \rho_u + (\mathbf{x}_f - \mathbf{x}_u)\cdot\nabla_u \rho$$

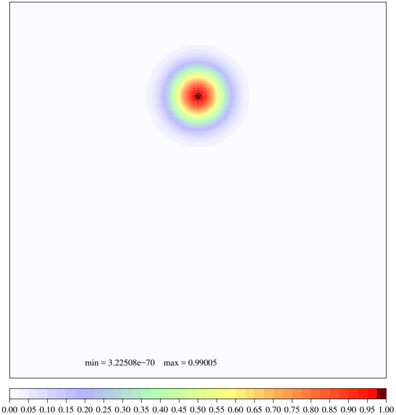
Results - Solid body rotation

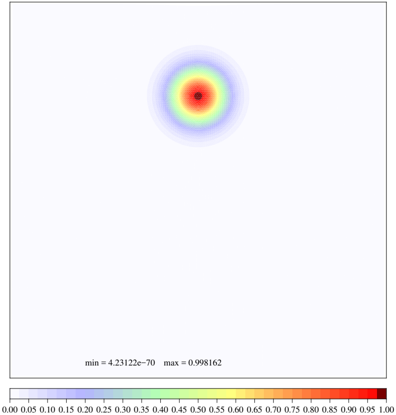
Test case from Leonard et al. [1996]

100×100 cells, 2,400 time-steps for one revolution

Fixed Mesh

monitor is smoothed $\min\left(1 + ||\nabla\nabla\rho|| \Big/ \overline{||\nabla\nabla\rho||},\ 4\right)$





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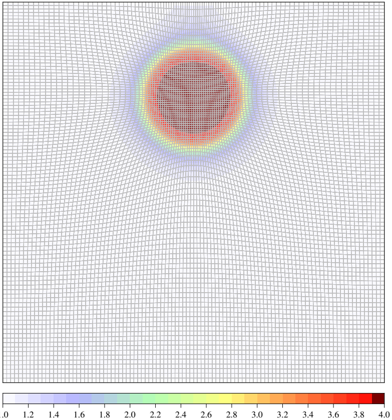

Advection over a Mountain and a Valley

Divergence free velocity field

100×100 cells, 2,400 time-steps for one revolution

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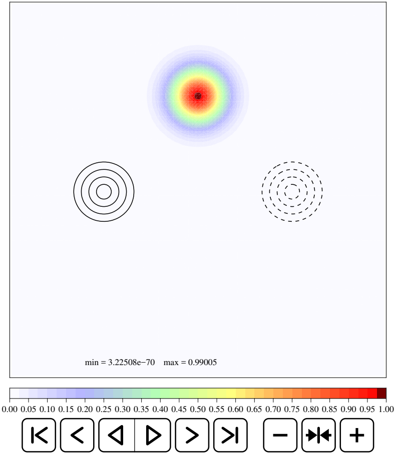

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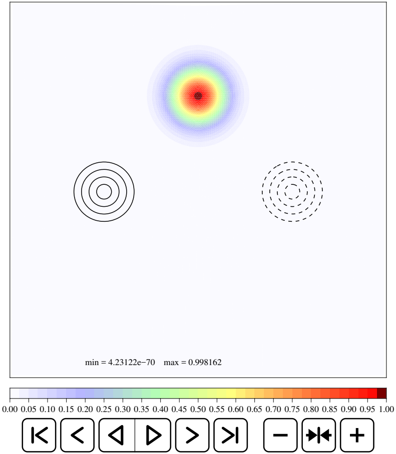

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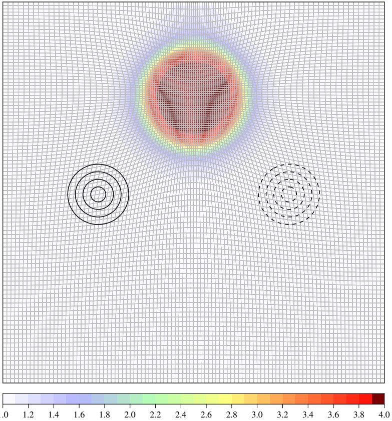
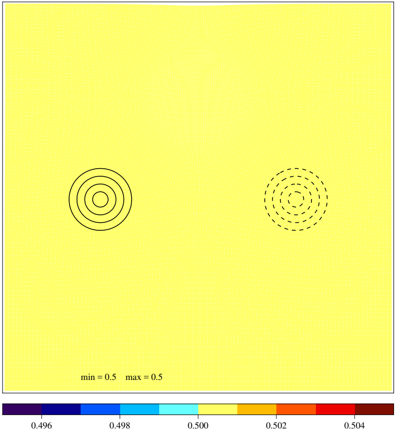
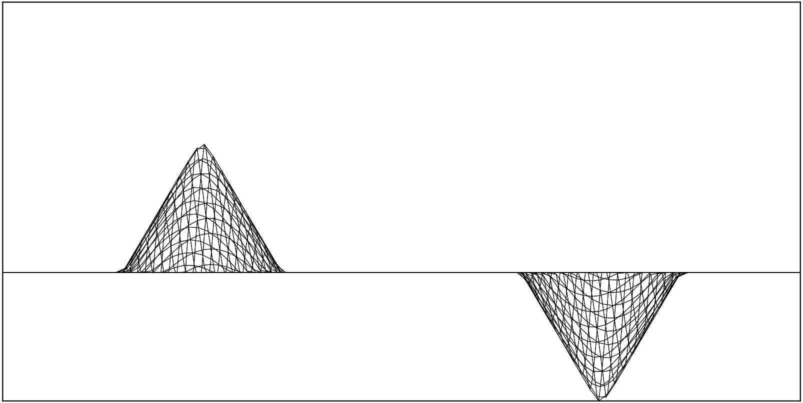


Advection of $\rho = 0.5$



Advection over a Mountain and a Valley

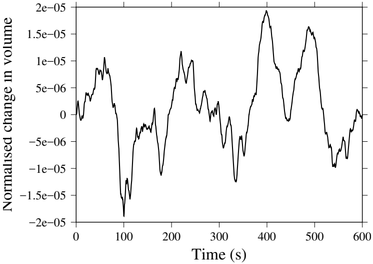
The volume of the domain changes as the resolution of the mountain changes
Cross section of the domain through the mountains



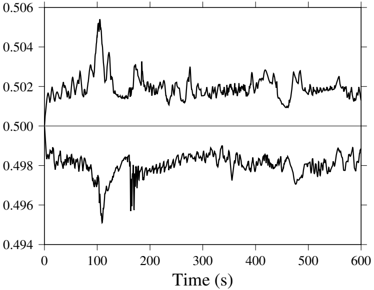
Advection over a Mountain and a Valley

ρ is conserved but it is squeezed and stretched as the domain changes

Change in Volume of the Domain

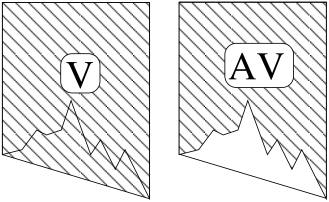


Minumum and maximum ρ



Calculate the Conserved Mesh Volume

V Cell volume from vertex locations
AV Real cell volume



- We do not want to do expensive conservative mappings to calculate AV every time step.
- But we know the mesh volume flux through every face each time step

$$\frac{A^{n+1}V^{n+1} - A^n V^n}{\Delta t} = \sum_{\text{faces}} A_f^n \phi_m^{n+1/2}$$

where $\phi_m = \mathbf{v} \cdot d\mathbf{S}$ is the mesh flux
A_f is A interpolated onto face f using upwind interpolation

Then solve the advection equation

$$\frac{A^{n+1}V^{n+1}\rho^{n+1} - A^n V^n \rho^n}{\Delta t} = -\sum_{\text{faces}} \rho_f A_f^n (\mathbf{u} \cdot d\mathbf{S} - \phi_m)$$

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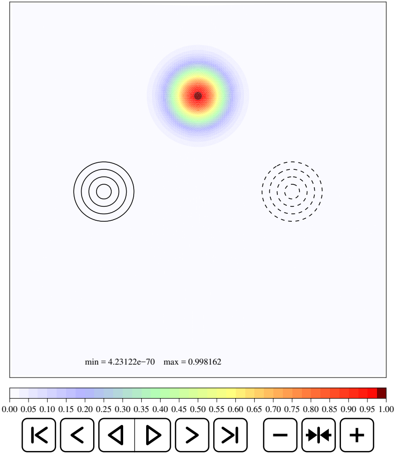
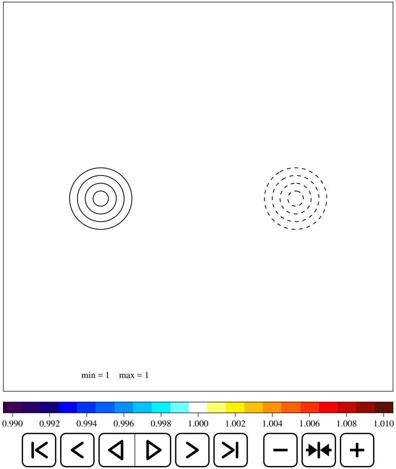
Also advection mesh volume A





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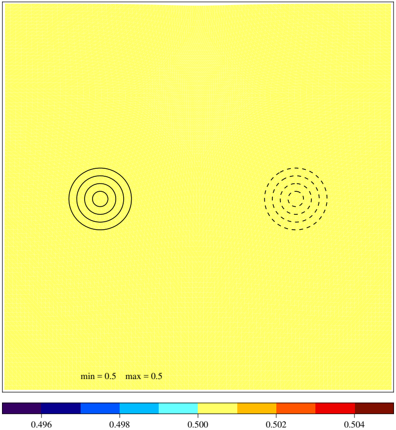
100×100 cells, 2,400 time-steps for one revolution

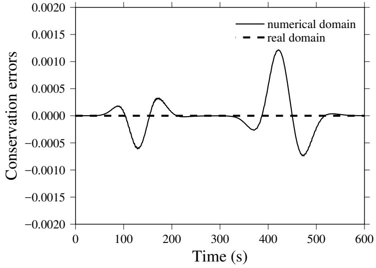
Advection of $\rho = 0.5$



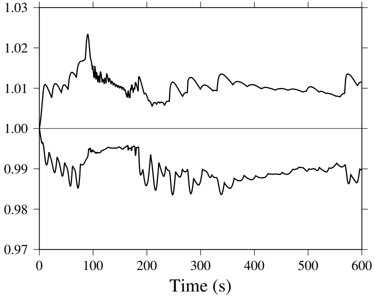
Advection over a Mountain and a Valley

ρ is conserved relative to the real domain size which is fixed

Conservation of ρ on the numerical domain and on the real domain



Minumum and maximum A



Shallow Water Equations

in a rotating frame:

$$
\begin{aligned}
\frac{\partial h\mathbf{u}}{\partial t} + \nabla \cdot (h\mathbf{u}\mathbf{u}) &= -2h\boldsymbol{\Omega} \times \mathbf{u} - gh\nabla h \\
\frac{\partial h}{\partial t} + \nabla \cdot (h\mathbf{u}) &= 0
\end{aligned}
$$

Definitions:

h	fluid depth
$\mathbf{u}$	fluid velocity
$\mathbf{u}_f$	velocity at face
g	gravity
$\boldsymbol{\Omega}$	rotation

OpenFOAM Co-located discretisation

- Apply Reynolds transport theorem to momentum in each cell of volume $\mathscr{V}(t)$:

$$\frac{(\mathscr{V}h\mathbf{u})^{n+1}-(\mathscr{V}h\mathbf{u})^{n}}{\Delta t}=-\int_S h\mathbf{u}\left(\mathbf{u}-\mathbf{v}\right)\cdot\mathbf{dS}-2\int_{\mathscr{V}}h\boldsymbol{\Omega}\times\mathbf{u}\,dV-\int_{\mathscr{V}}gh\nabla h\,dV$$

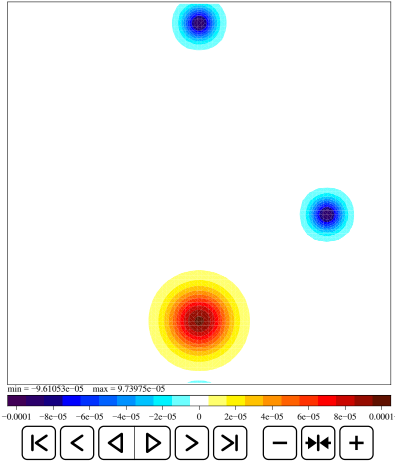
- Semi-implicit Crank-Nicolson time-stepping with off-centering 3/5 and deferred correction of explicit terms (superscript ℓ):

$$\begin{aligned}(\mathscr{V}h\mathbf{u})^{n+1} &= (\mathscr{V}h\mathbf{u})^{n}\\ &- \frac{2\Delta t}{5}\left\{\int_S h\mathbf{u}\left(\mathbf{u}-\mathbf{v}\right)\cdot\mathbf{dS}-2\int_{\mathscr{V}}h\boldsymbol{\Omega}\times\mathbf{u}\,dV-\int_{\mathscr{V}}gh\nabla h\,dV\right\}^{n}\\ &- \frac{3\Delta t}{5}\left\{\int_S h\mathbf{u}^{\ell}\left(\mathbf{u}-\mathbf{v}\right)^{\ell}\cdot\mathbf{dS}-2\int_{\mathscr{V}}h^{\ell}\boldsymbol{\Omega}\times\mathbf{u}^{\ell}\,dV-\int_{\mathscr{V}}gh^{\ell}\nabla h^{n+1}\,dV\right\}\end{aligned}$$

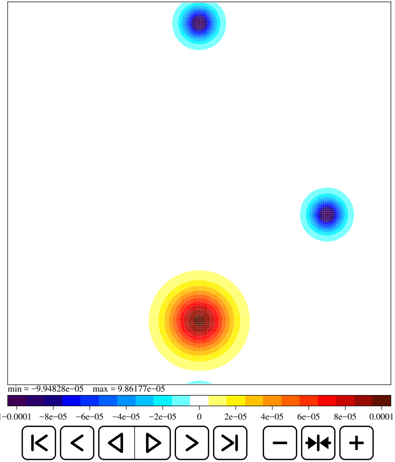
- Compact Helmholtz equation to calculate h for accurate wave dispersion

Interacting Vortices on a doubly periodic f-plane

100×100 uniform mesh



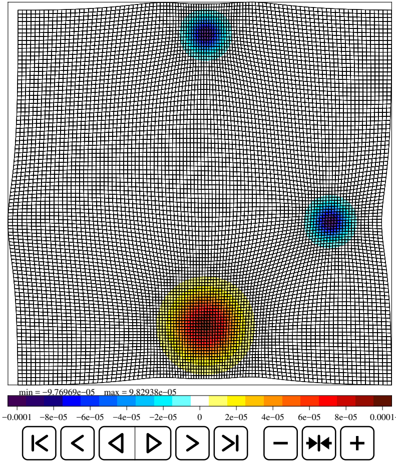
200×200 uniform mesh



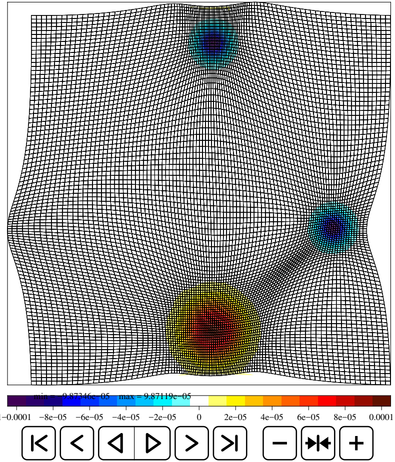
Monitor function based on gradient of vorticity

100×100, monitor Function $=$

$$\min\left(1 + |\nabla\xi| / \overline{|\nabla\xi|}\,, 2\right)$$



$$\min\left(1 + |\nabla\xi| / \overline{|\nabla\xi|}\,, 4\right)$$



Costs and time-steps

	Max cell area ratio	Δt (s)	Δx (km) min	Δx (km) max	Max Courant	CPU time (s)
100×100	1	250	180	180	0.1	2509
200×200	1	250	90	90	0.2	7044
100×100	2	250	~ 140	~ 200	0.13	3852
100×100	4	250	~ 106	~ 213	0.17	3773

Conclusions

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Conclusions

- Equation of Monge-Ampère type solved on the surface of a sphere for generating an optimally transported mesh
- Advection over a mountain using a moving mesh
  - Accuracy gains using adaptivity
  - We have achieved exact conservation and preservation of uniform fields over mountains without conservative remapping
- Shallow water equations on moving meshes
  - Solving Monge-Ampère equation is $< \frac{1}{3}$ of total cost
  - Accuracy gains using adaptivity
  - Distorted meshes can lead to spurious solutions

References

C.J. Budd, R.D. Russell, and E. Walsh. The geometry of r-adaptive meshes generated using optimal transport methods. *J. Comput. Phys.*, 282:113–137, 2015.

B.H. Fiedler and R. J. Trapp. A fast dynamic grid adaption scheme for meteorological flows. *Mon. Wea. Rev.*, 121(10):2879–2888, 1993.

C. Kühnlein, P.K. Smolarkiewicz, and A. Dörnbrack. Modelling atmospheric flows with adaptive moving meshes. *J. Comput. Phys.*, 231(7):2741–2763, 2012.

B.P. Leonard, A.P. Lock, and M.K. MacVean. Conservative explicit unrestricted-time-step multidimensional constancy-preserving advection schemes. *Mon. Wea. Rev.*, 124(11):2585–2606, 1996.

H. Weller, P. Browne, C. Budd, and M. Cullen. Mesh adaptation on the sphere using optimal transport and the numerical solution of a Monge-Ampère type equation. *J. Comput. Phys.*, 308:102–123, 2016.

Shang-Ping Xie, Clara Deser, Gabriel A. Vecchi, Matthew Collins, Thomas L. Delworth, Alex Hall, Ed Hawkins, Nathaniel C. Johnson, Christophe Cassou, Alessandra Giannini, and Masahiro Watanabe. Towards predictive understanding of regional climate change. *Nature Clim. Change*, 2015.