Available Potential Energy and Exergy in Stratified Fluids

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Abstract

Lorenz’s theory of available potential energy (APE) remains the main framework for studying the atmospheric and oceanic energy cycles. Because the APE generation rate is the volume integral of a thermodynamic efficiency times the local diabatic heating/cooling rate, APE theory is often regarded as an extension of the theory of heat engines. Available energetics in classical thermodynamics, however, usually relies on the concept of exergy and is usually measured relative to a reference-state maximizing entropy at constant energy, whereas APE’s reference state minimizes potential energy at constant entropy. This review seeks to shed light on the two concepts; it covers local formulations of available energetics, alternative views of the dynamics/thermodynamics coupling, APE theory and the second law of thermodynamics, APE production/dissipation, extensions to binary fluids, mean/eddy decompositions, APE in incompressible fluids, APE and irreversible turbulent mixing, and the role of mechanical forcing on APE production.
1. INTRODUCTION

The concept of available potential energy (APE) was first introduced in the atmospheric context by Margules (1905) (who termed it “available kinetic energy”) and subsequently by Lorenz (1955). Margules and Lorenz sought to understand the source of energy for storms and the nature of the processes maintaining the global large-scale atmospheric circulation against dissipation, respectively. They both recognized that only a fraction of the total potential energy (the sum of the internal and gravitational potential energies) is actually available for conversion into kinetic energy. They defined APE as the difference of potential energy between the actual state and the reference-state minimizing potential energy in an isentropic rearrangement of mass, as illustrated in Figure 1. Dutton & Johnson (1967), Van Mieghem (1973), Dutton (1986), and Wiin-Nielsen & Chen (1993) have reviewed and extensively discussed APE and atmospheric energetics.

For a hydrostatic atmosphere, APE is commonly expressed as

\[ \text{APE} := \int_V [b - b_R] \, dm, \]  

(1)

where \( b \) and \( b_R \) are the specific enthalpies of the actual and reference state, respectively, and \( dm = \rho dV \) is the mass of an elementary air parcel (Pauluis 2007). Treating the atmosphere as a dry gas leads to the following two-component energy cycle:

\[ \frac{dK}{dr} = C(A, K) - D, \]  

(2)

\[ \frac{dA}{dr} = -C(A, K) + \int_V \left( \frac{T - T_R}{T} \right) \dot{Q} \, dm, \]  

(3)

where \( K \) is the volume-integrated kinetic energy, \( A \) is the total APE, \( C(A, K) \) is the energy conversion between APE and the kinetic energy, \( D \) is the volume-integrated viscous dissipation, and \( T_R \) is the temperature of the parcel in the reference state. A key feature of Equation 3 is the APE generation term \( G \) having the form of a volume integral of a thermodynamic efficiency-like factor \( (T - T_R)/T \) times the local diabatic heating/cooling rate \( \dot{Q} \), which is reminiscent of the celebrated Carnot (1824) formula and suggests a link with the classical theory of heat engines.

This link is not entirely clear, however, because the concept of APE differs from the concept of available energy that had historically been introduced as part of the development of heat engines by Thomson (1849), Maxwell (1871), and Gibbs (1873, 1875), among others, now commonly referred...
to as exergy, after Rant (1956) (see also Bejan 1997 for a historical respective on the development of these ideas). Indeed, exergy usually measures the available thermodynamic energy arising from the departure of the system considered from thermodynamic (and mechanical) equilibrium, and hence from a reference state maximizing entropy at constant energy. In contrast, the reference state entering Margules’s/Lorenz’s APE minimizes potential energy at constant temperature, which is fundamentally different. The apparent disagreement between Margules’s/Lorenz’s APE theory and the classical thermodynamic concept of exergy prompted a number of studies arguing that available energetics in the atmosphere should be based on an isothermal reference state, presumably the state of maximum entropy (e.g., Dutton 1973, Livesey & Dutton 1976, Pichler 1977, Pearce 1978, Blackburn 1983, Karlsson 1990, Marquet 1991, Bannon 2005). As shown by many authors, the use of exergy-based available energetics leads to relatively straightforward formulations of local available energetics, the lack of which in Lorenz’s APE theory had impeded its application to the study of energetics in regional domains. Andrews’ (1981) study marked a breakthrough in the field, showing for the first time how Lorenz’s globally defined APE could in fact be derived from a local APE density. This allowed Marquet (1995) and Kucharski (1997, 2001) to greatly clarify the links between APE and exergy. This review surveys the main conceptual issues underlying various theories of available energetics in stratified fluids, with a number of illustrations drawn from some recent applications.

2. THERMODYNAMIC AND FLUID DYNAMICS VIEWS OF AVAILABLE ENERGY

2.1. Useful Work, Exergy, and the Convexity of Internal Energy

One of the main concerns of classical thermodynamics is understanding the general principles controlling the amount of useful work that can be produced by devices exchanging heat and work with an environment of much larger dimensions, thus often idealized as an isothermal reservoir of uniform temperature $T_0$ and pressure $P_0$. Owing to its importance in a considerable number of fields of physics, the issue has given rise to a large body of literature. Useful reviews and discussions of the main results have been provided by Keenan (1951), Haywood (1974), and Gaggioli (1998), whereas Karlsson (1990) and Marquet (1991) also reviewed historical developments. It seems now relatively well understood that the main thermodynamic property underlying available energy or exergy is the convexity of the specific internal energy $e = e(\eta, \upsilon)$ in the (specific entropy $\eta$, specific volume $\upsilon$) space. Convexity ensures that for any convex function $f(x)$, the function $f_{ex}(x)$,

$$f_{ex}(x) := f(x) - \nabla_x f(x_0) \cdot (x - x_0) \approx \frac{1}{2}(x - x_0)^T H(x_0)(x - x_0), \quad (4)$$

is sign positive definite, where $H(x_0)$ is the Hessian matrix of the second derivatives at $x = x_0$. The application of such a construction to internal energy, whose total differential is $de = T d\eta - P d\upsilon$, with $T$ and $P$ the temperature and pressure, respectively, defines the exergy of internal energy:

$$e_{ex} := e - e_0 - T_0(\eta - \eta_0) + P_0(\upsilon - \upsilon_0). \quad (5)$$

The associated geometrical construction is illustrated in Figure 2. The positive-definite character of $e_{ex}$ is easily established by decomposing Equation 5 into work and heat components $e_{ex} = e_{ex}^{work} + e_{ex}^{heat}$:

$$e_{ex}^{work} := b - b(\eta, P_0) + (P_0 - P)\upsilon = -\int_{P_0}^{P} \int_{\rho}^{\rho'} \frac{1}{\rho^2 c_s^2} dP' d\rho', \quad (6)$$

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$$e_{ex}^{work} := b - b(\eta, P_0) + (P_0 - P)\upsilon = -\int_{P_0}^{P} \int_{\rho}^{\rho'} \frac{1}{\rho^2 c_s^2} dP' d\rho', \quad (6)$$
Figure 2
Schematics illustrating the convex character of the internal energy in the specific entropy/volume space. The parabolic-shaped surface represents a surface of constant internal energy, whereas the plane surface represents the plane tangent to the internal energy surface passing through the reference value \((\eta_R, v_R, e(\eta_R, v_R))\). Internal energy is convex because its surface always lies on one side of its tangent plane so that the exergy is defined as the volume lying between the internal energy surface and the tangent plane, as per Equation 4.

\[
e_{\text{heat}} = b(\eta, P_0) - b(\eta_0, P_0) - T_0(\eta - \eta_0) = \int_{\eta_0}^{\eta} \int_{\eta_0}^{\eta'} \frac{T}{c_p} \, d\eta'' \, d\eta',
\]

which clearly demonstrates that the exergy of internal energy resides in the departure of the system from mechanical and thermodynamical equilibrium. In the thermodynamics literature, the quadratic quantity in Equation 8 is often called the thermodynamic length and plays an important role in thermodynamic optimization theory (e.g., Salamon & Berry 1983, Crooks 2007). The anergy is defined as the difference between the internal energy and exergy:

\[
a := e - e_{\text{ex}} = e_0 + T_0(\eta - \eta_0) - P_0(v - v_0).
\]

The above decomposes the internal energy as the sum of the anergy \(a\) and the sign positive-definite work and heat exergies, \(e_{\text{work}}^+ + e_{\text{heat}}^+\). These three different quantities have a well-defined local meaning, even for temporally and spatially varying \(T_0\) and \(P_0\), and can therefore be used to describe local energetics. Although the most common choice in classical thermodynamics is to use constant \(T_0\) and \(P_0\), many authors have proposed the use of a hydrostatically balanced \(P_0\) to apply the concept to meteorology (e.g., Fortak 1998, Bannon 2005). Instead of using a
constant $T_0$, some authors have suggested that $T_0$ should be the temperature of the isothermal state corresponding to a state of maximum entropy (e.g., Dutton 1973, Livesey & Dutton 1976). In that case, exergy measures the distance to a state in both thermodynamic and mechanical equilibrium, with $T_0 = T_0(t)$ a function of time, and $P_0 = P_0(z, t)$ a function of both $z$ and $t$. The corresponding local formulation of energetics is as follows:

$$\rho \frac{D}{Dt} \left[ \frac{v^2}{2} + e_{ex} \right] + \nabla \cdot \left[ (\rho P_0) v \right] = \rho G_0 + \rho v \cdot F_v,$$

(10)

$$\rho \frac{D}{Dt} (a + \Phi) + \nabla \cdot (P_0 v) = \rho T \frac{D\eta}{Dt} - \rho G_0,$$

(11)

where the generation term is given by

$$\rho G_0 = \left( \frac{T - T_0}{T} \right) \dot{Q} + \rho (\eta_0 - \eta) \frac{dT_0}{dt} + \left( 1 - \frac{\rho}{\rho_0} \right) \frac{\partial P_0}{\partial t},$$

(12)

where $\dot{Q} = \rho T \frac{D\eta}{Dt}$ is the local total diabatic heating/cooling rate. In writing the above equations, we assume that the momentum and kinetic energy equations take the following forms:

$$\rho \frac{Dv}{Dt} + 2 \Omega \times v + \nabla P = -\rho \nabla \Phi + \rho F_v,$$

(13)

$$\rho \frac{Dv^2}{2} + \nabla \cdot (Pv) = \rho P \frac{Dv}{Dt} - \rho \frac{D\Phi}{Dt} + \rho v \cdot F_v,$$

(14)

where $v = (u, v, w)$ is the three-dimensional velocity field, $\Omega$ is Earth’s rotation vector, $\Phi$ is the geopotential, and $F_v$ is a representation of the viscous force. The term $C(e_{ex}, e_t)$ represents the conversion between exergy and kinetic energy.

The quantities $v^2 + e_{ex}$ and $a + \Phi$ obey separate conservation laws in the absence of diabatic/viscous effects but become coupled through the generation term $G_0$ (Equation 12) when diabatic/viscous effects are retained. When $T_0$ and $P_0$ are chosen to be time independent, $G_0$ reduces to a Carnot-like thermodynamic production term, with a local thermodynamic efficiency $(T - T_0)/T$. When $P_0$ is chosen to be spatially uniform, $e_{ex}^{\text{work}}$ and $e_{ex}^{\text{rest}}$ are decoupled in the absence of diabatic/viscous effects, but the use of a $z$-dependent $P_0(z, t)$ couples heat and work even for purely adiabatic/inviscid motions, making such decomposition less meaningful. Such a coupling can be avoided for $P_0 = P_0(\eta)$, a function of entropy alone, but this would be at the expense of satisfying hydrostatic balance. Equations similar to Equations 10 and 11 have been discussed in great detail by Fortak (1998), who examined how to define exergies and anergies for the most common thermodynamic potentials (see also Bannon 2005 for a generalization of exergy to multicomponent fluids, with an application to a moist atmosphere, where $P_0$ also satisfies hydrostatic balance).

### 2.2. Extended Exergy and APE Density for Stratified Fluids

Although the above exergy and anergy can obviously be used to describe local energetics, a significant departure from thermodynamic equilibrium may, however, leave a large fraction of the exergy untapped and effectively unavailable. Similarly, the local thermodynamic efficiency $(T - T_0)/T$ may significantly overestimate the local production of kinetic energy due to diabatic heating/cooling, as a thermodynamic efficiency $(T - T_0)/T$ constructed from the $z$-dependent background temperature $T_R(z)$ would be on average smaller. Simply replacing the constant or
time-dependent $T_0$ by a $z$-dependent $T_R(z, t)$ in Equation 5 is not satisfactory, however, because, as shown by Kucharski (1997, 2001), this couples $\nu^2/2 + e_{ex}$ and $a + \Phi$ even for purely adiabatic motions. Kucharski (1997, 2001) showed that to avoid such a coupling, the energy and exergy have to be generalized as follows:

$$a := c_R + Q(\eta, S) - Q(\eta_R, S_R) - P_R(v - v_R),$$  \tag{15}$$

$$e_{ex} := e - a = e - c_R - Q(\eta, S) + Q(\eta_R, S_R) + P_R(v - v_R),$$  \tag{16}$$

where $Q(\eta, S)$ is a function of the materially conserved variables $\eta$ and $S$ (discussed below), where $S$ can represent either salinity or the total water mixing ratio, depending on the context (Kucharski’s results were derived for a one-component fluid; we extend them here for a binary fluid). Salinity is the ratio of the mass of dissolved salts per unit mass of seawater (i.e., freshwater plus the dissolved salts). The total water mixing ratio is the ratio of the mass of water species (vapor, liquid, and frozen) per unit mass of dry air. It is clear that Equations 15 and 16 encompass the previous definitions in Equations 5 and 9, which are recovered in the particular case in which $Q(\eta, S) = Q(\eta) = T_0\eta$.

An explicit construction of the function $Q(\eta, S)$ entering Equations 15 and 16 was first discussed by Andrews (1981) in the case of a one-component fluid in the absence of diabatic and viscous effects. By considering a hydrostatically balanced reference state with $z$-dependent reference entropy and pressure profiles $\eta_R(z)$ and $P_R(z)$, such that $\eta_R/dz > 0$ everywhere, one can use entropy as a vertical coordinate and thus regard $P_R = P_R(\eta_R)$ as a function of specific entropy. In that case, it is possible to express the reference temperature profile as $T_R(z, P_R(z)) = T(\eta_R, P_R(\eta_R)) = \hat{T}(\eta_R)$ and to construct the quantity $Q(\eta)$ from

$$Q(\eta) - Q(\eta_R) = \int_{\eta_R}^{\eta} T_R(\eta', P_R(\eta'))d\eta' = \int_{\eta_R}^{\eta} \hat{T}(\eta')d\eta'.$$  \tag{17}$$

Andrews (1981) was the first to show that in the case to that corresponds to that introduced by Margules and Lorenz, the quantity $e_{ex}$ (Equation 16) can be regarded as the local counterpart of Lorenz’s globally defined APE, which he called APE density. Shepherd (1993) recovered the same quantity from a Hamiltonian approach and called it the pseudo-energy, whereas Kucharski (1997) called it the extended exergy. For $Q$ defined by Equation 17, the sign positive-definite character of $e_{ex}$ can be established from the decomposition $e_{ex} = e_{ex}^{work} + e_{ex}^{heat}$ in terms of the following pressure and entropy components:

$$e_{ex}^{work} := b - b(\eta_R, P_R) + (P_R - P)v = -\int_{P_R}^{P} \int_{\rho}^{\rho'} \frac{1}{\rho' c_P} dP' d\rho',$$  \tag{18}$$

$$e_{ex}^{heat} := b(\eta, P_R) - b(\eta_R, P_R) - \int_{\eta_R}^{\eta} \hat{T}(\eta')d\eta' = \int_{\eta_R}^{\eta} \int_{\eta'}^{\eta_R} \Gamma(\eta', P_R(\eta')) dP_R d\eta' d\eta',$$  \tag{19}$$

where $\Gamma = \frac{\partial^2 b}{\partial \eta \partial P} = \alpha T/(\rho c_P)$ is the adiabatic lapse rate, with $\alpha$ the thermal expansion coefficient. The linearized expression of the pressure component is the same as that for classical exergy. The linearized expression for the entropy component of extended exergy becomes

$$e_{ex}^{heat} \approx -\Gamma_R \frac{dP_R}{d\eta_R} (\eta - \eta_R)^2 = -\Gamma_R \frac{\partial P_R}{\partial \eta} \frac{(\eta - \eta_R)^2}{2\partial \eta_R/\partial z} = \frac{1}{2} N_{\rho_k}^2 \eta_z^2, $$  \tag{20}$$

where $N_{\rho_k}^2 = \rho_k G \beta \partial \eta_R/\partial z$, as the reference pressure is in hydrostatic balance, and $\partial P_k/\partial z = -\rho_k g$, as the reference state $\eta(x, t) = \eta_R(z_R, t)$ by construction, with $z = z - z_R$ being the vertical parcel displacement from its reference position.
In the general case of a binary fluid in which the reference state can be altered by diabatic effects, the various reference profiles must be regarded as functions of time as well. Although the density and pressure \( \rho_k = \rho_k(z, t) \) and \( P_k = P_k(z, t) \) must remain functions of \( z \) and \( t \) and in hydrostatic balance at all times, it seems possible for the reference entropy and temperature profiles to possess horizontal variations provided that they are density compensated, i.e., that they satisfy \( \rho_k(\eta_k(x, t), S_k(x, t), P_k(z, t)) = \rho_k(z, t) \). In that case, the reference pressure can also be regarded as \( P_k = P_k(z, t) = P_k(\eta_k, S_k, t) \), a function of \( \eta_k, S_k, \) and \( t \). In the most general case, therefore, \( Q(\eta, S, t) \) becomes a function of entropy, \( S \), and \( t \) and can be obtained from integrating the following differential relation: \( dQ = T(d\eta + \mu dS) \), where \( \mu \) is the relative chemical potential, \( T(\eta, S, t) = T(\eta, S, P_k(\eta, S, t)) \), and \( \mu(\eta, S, t) = \mu(\eta, S, P_k(\eta, S, t)) \). The corresponding evolution equations for \( \nabla^2 + \epsilon_{ex} \) and \( \alpha + \Phi \) therefore become

\[
\rho \frac{D}{Dt} \left( \frac{\nabla^2}{2} + \epsilon_{ex} \right) + \nabla \cdot [(P - P_k)\mathbf{v}] = \rho G_{ex} + \rho \mathbf{v} \cdot \mathbf{F}_e, \tag{21}
\]

\[
\rho \frac{D}{Dt} [\alpha + \Phi] + \nabla \cdot (P_k\mathbf{v}) = \rho \left[ T \frac{D\eta}{Dt} + \mu \frac{DS}{Dt} \right] - \rho G_{ex}, \tag{22}
\]

where the local APE generation term is given by

\[
\rho G_{ex} = \rho \left\{ (T - T_k) \frac{D\eta}{Dt} + (\mu - \tilde{\mu}) \frac{DS}{Dt} + \frac{\partial(Q_k - Q)}{\partial t} \frac{\partial P_k}{\partial t} \right\} + \left( 1 - \frac{\rho}{\rho_k} \right) \frac{\partial P_k}{\partial t}. \tag{23}
\]

The expressions in Equations 21–23 mimic the corresponding expressions for the classical exergy (Equations 10–12). Our approach to a binary fluid is somewhat more general than Bannon’s (2004), which considered only \( z \)-dependent \( \eta_k(z) \) and \( S_k(z) \) reference profiles. The expression is valid for nonhydrostatic motions. These equations show that in the absence of diabatic and viscous effects, \( \alpha + \Phi \) and \( \nabla^2 + \epsilon_{ex} \) are individually conservative quantities. It is important to note that locally, the generation term (Equation 23) possesses a number of additional terms, namely the pressure and \( Q \) terms, that are lacking in Pauluis’s (2007) global APE approach. Scotti et al. (2006) and Molemaker & McWilliams (2010) provided indications, in the context of Boussinesq fluids, that these additional terms may occasionally dominate and therefore be key to understanding local kinetic energy production.

Kucharski & Thorpe (2000) used the extended exergy to describe the energy cycle of an idealized baroclinic wave. Figure 3 shows that in general, the distribution of extended exergy or APE density differs significantly from that of classical exergy, and likewise for the corresponding Carnot-like thermodynamic efficiency factors. The difference in thermodynamic efficiency is most dramatically illustrated in the oceanic case; indeed, Tailleux (2010) showed that whereas the classical exergy predicts a net production of exergy \( O(90 \text{ TW}) \) owing to surface buoyancy fluxes, this number reduces to \( O(0.5 \text{ TW}) \) when APE is used, which is about two orders of magnitude smaller, a considerable difference!

### 2.3. Alternative Views on the Dynamics/Thermodynamics Coupling

Although APE theory emphasizes the construction of a positive-definite potential energy reservoir, the description of the atmospheric and oceanic energy cycles, however, is concerned with the magnitude and sign of the energy conversions, rather than with the sign of the reservoirs. From that viewpoint, the construction of Lorenz’s APE is significant only to the extent that it helps anticipate the sign and magnitude of the kinetic-to-potential energy conversion. To shed light on the issue, we review a number of approaches providing an alternative view of the thermodynamics/dynamics coupling that take as their starting point the following local evolution equation for the kinetic energy production:

\[
\frac{D}{Dt} \left( \frac{\mathbf{v}^2}{2} + \epsilon_{ex} \right) + \nabla \cdot [(P - P_k)\mathbf{v}] = \rho G_{ex} + \rho \mathbf{v} \cdot \mathbf{F}_e.
\]
Figure 3
Difference between the extended exergy [or available potential energy (APE) density, or pseudo-energy] and classical exergy for an idealized atmospheric temperature distribution (top panels), and the corresponding thermodynamic efficiencies (bottom panels). Figure adapted with permission from Kucharski (1997).

\[
\rho \frac{D}{Dt} \frac{v^2}{2} = -\rho \frac{D\Phi}{Dt} - v \cdot \nabla P + \rho v \cdot F_v.
\]  
(24)

Instead of writing down the pressure work as \(v \cdot \nabla P = \nabla \cdot (P v) - \rho P \frac{Dv}{Dt}\) to make the link with internal energy, one can alternatively regard the pressure work as a conversion with the specific enthalpy \(b\) (e.g., Dutton 1992), which has a total differential of \(db = T \frac{D\eta}{Dt} + \mu \frac{DS}{Dt} + v \frac{Dp}{Dt}\) so that

\[
v \cdot \nabla P = \frac{DP}{Dt} - \frac{\partial P}{\partial t} = \rho \left[ T \frac{D\eta}{Dt} - \mu \frac{DS}{Dt} \right] - \frac{\partial P}{\partial t}.
\]  
(25)

This leads to a description of local energetics that takes the generic form

\[
\rho \frac{De_k}{Dt} = -C(e_k, \epsilon_b) - C(e_k, \epsilon_g) + \rho v \cdot F_v,
\]  
(26)

\[
\rho \frac{De_b}{Dt} = C(e_b, \epsilon_b) + \rho G_b,
\]  
(27)

\[
\rho \frac{De_g}{Dt} = C(e_g, \epsilon_g).
\]  
(28)
where $e_k = v^2/2$, $e_g = \Phi$, and the energy conversions are $C(e_k, e_g) = v \cdot \nabla P$ and $C(e_k, e_g) = \rho D\Phi/ Dt$. If $e_k$ is taken as the specific enthalpy, the generation term $G_h$ is given by

$$ \rho G_h = \rho \left[ T \frac{D\eta}{Dt} + \mu \frac{DS}{Dt} \right] + \frac{\partial P}{\partial t}. \quad (29) $$

A simple analysis of the structure of Equations 27 and 28 reveals, however, that it is possible to obtain alternative formulations by defining $e^*_h = e_h - Q(\eta, S)$, where $Q(\eta, S)$ is an arbitrary function of the materially conserved variables $\eta$ and $S$, such that

$$ \rho D\frac{e^*_h}{Dt} = C(e_k, e^*_h) + G^*_h, \quad (30) $$

$$ \rho G^*_h = \rho \left[ (T - T^*) \frac{D\eta}{Dt} + (\mu - \mu^*) \frac{DS}{Dt} \right] + \frac{\partial P}{\partial t}. \quad (31) $$

by defining $T^* = \partial Q/\partial \eta$ and $\mu^* = \partial Q/\partial S$. Clearly, $C(e_k, e_h) = C(e_k, e^*_h)$, so the transformation leaves the energy conversion between the kinetic energy and thermodynamic energy unchanged.

2.3.1. Example 1: Marquet’s (1991) available enthalpy. Motivated by Pearce (1978) and classical exergy, Marquet (1991) introduced the available enthalpy $e^*_h = h - T_0 \eta$ for a dry atmosphere, corresponding to the choice $Q(\eta, S) = T^\eta_0$, with $T_0$ constant, in which case $G^*_h$ becomes

$$ \rho G^*_h = \rho (T - T_0) \frac{D\eta}{Dt} + \frac{\partial P}{\partial t}. \quad (32) $$

An interesting property of available enthalpy is that for a perfect gas, it naturally splits into temperature and pressure components $e^*_h = aT + aP$, with

$$ a_T = e_p(T - T_0) - e_p T_0 \ln \frac{T}{T_0}, \quad a_P = RT_0 \ln \frac{P}{P_0}, $$

where $a_T$ is sign positive definite. Although $a_P$ is not sign definite, it appears nevertheless possible to obtain a meaningful and rigorous decomposition of available enthalpy into mean and eddy components, such as

$$ a_T = e_p(T - T^*) - e_p T^* \ln \frac{T}{T^*} + e_p(T^* - T_0) - e_p T_0 \ln \frac{T^*}{T_0}, \quad (eddy) $$

$$ a_P = RT_0 \ln \frac{P^*}{P_0} + RT_0 \ln \frac{P^*}{P_0} \quad (mean) $$

if $T^*$ and $P^*$ are taken as mean properties, depending solely on height, for instance, defining the eddy parts $T^e = T - T^*$ and $P^e = P - P^*$. In fact, the properties of $a_T$ and $a_P$ are such that a decomposition in an arbitrary number of subcomponents is possible without introducing any interaction term, which normally plagues classical eddy/mean decompositions. This framework was used by Marquet (2003) to diagnose a baroclinic wave energy cycle, and extensions to a moist atmosphere were discussed by Marquet (1993) and Bannon (2005).

2.3.2. Example 2: dynamic/potential enthalpy decomposition. Another important construction is based on using the potential enthalpy $Q = b(\eta, S, P_0)$ (e.g., McHall 1990, McDougall 2003), which McDougall (2003) argued in the oceanic context is the most accurate quantity to
measure “heat,” where $P_0$ is a fixed reference pressure. This leads one to define $e^*_h$ as

$$e^*_h = b - b(\eta, S, P_0) = \int_{P_0}^P v(\eta, S, P') \, dP',$$

which was termed the dynamic enthalpy by Young (2010) and the effective potential energy by Nycander (2010), and the generation term as

$$\rho G^*_h = \rho \left( (T - \theta) \frac{D\eta}{Dt} + (\mu - \mu_r) \frac{DS}{Dt} \right) + \frac{\partial P}{\partial t}.$$

where $\theta$ is the potential temperature, defined by the implicit relationship $\eta(T, S, P) = \eta(\theta, S, P_0)$ in the oceans, and $\mu_r = \mu(\eta, S, P_0)$. In the absence of diabatic effects, when $\partial P/\partial t$ is small enough to be neglected, the standard Bernoulli theorem stating that $B = v^2/2 + \Phi + \int_{P_0}^P v(\eta, S, P) \, dP$ is conserved along streamlines is recovered. In the oceans, because $T = \theta$ and $\mu = \mu_r$ at the ocean surface where $P = P_0$, the volume integral of $G^*_h$ has an interesting property: It lacks any dependence on the surface buoyancy fluxes and is entirely controlled by irreversible molecular diffusive processes. Recently, the global budgets of dynamic and potential enthalpy have been shown to play a key role in extending Paparella & Young’s (2002) epsilon theorem to a fully compressible ocean with a general nonlinear equation of state (see Tailleux 2012b).

2.3.3. Example 3: link with APE density approach. Finally, an obvious choice motivated by the construction of the APE density is $e^*_h = b - Q(\eta, S)$, with $Q(\eta, S)$ defined from Lorenz’s reference state, as explained in Section 2.3, for which the generation term becomes

$$\rho G^*_h = \rho \left( (T - \tilde{T}) \frac{D\eta}{Dt} + (\mu - \tilde{\mu}) \frac{DS}{Dt} \right) + \frac{\partial P}{\partial t}.$$

Interestingly, even though $e^*_h$ is neither sign positive definite nor vanishing for Lorenz’s reference state, $G^*_h$ is nevertheless very close to $G_e$ (Equation 23) from which it differs only in the pressure-dependent term. As a result, it has a similar predictive power as $G_e$ with regard to anticipating the local production of kinetic energy by diabatic effects.

2.4. APE Density in Incompressible Boussinesq Fluids

As seen above, the APE density for a compressible stratified fluid appears to be rooted in the convexity of internal energy and hence fundamentally a thermodynamic quantity (without being a function of state, however). This is intriguing because atmospheric and oceanic motions are often regarded as well described by the incompressible Boussinesq approximation, for which the globally defined APE is usually defined only in terms of gravitational potential energy, e.g.,

$$\text{APE} = \int_V \rho g(z - z_h(x, t)) \, dV.$$

Although it might be tempting to conclude that APE is of a different nature in a Boussinesq fluid, it is important to realize that the nature of a quantity should not depend on the kind of approximations used to describe it. For reassurance that APE is indeed a thermodynamic quantity even in Boussinesq fluids, however, one needs to invoke Holliday & McIntyre’s (1981) results, which demonstrate that Equation 36 can actually be regarded as the volume integral of the following sign positive-definite APE density $E_a$:

$$E_a(x, t) = \rho g(z - z_h(x, t)) + P_h(z, t) - P_h(z_h(x, t), t).$$

where $P_h(z, t)$ is the reference pressure in hydrostatic balance with the reference density $\rho_h(z, t)$. By using the property that $\rho(z, t) = \rho_h(z_h, t)$, which defines the reference depth $z_h(x, t)$, one can
show that Equation 37 is equivalent to

$$E_a = -\int_0^\zeta g \tilde{\zeta} \rho' (z - \tilde{\zeta}) \, d\tilde{\zeta} \approx \rho_0 N^2_h \frac{\zeta^2}{T},$$  \hspace{1cm} (38)$$

where $N^2_h = -(g/\rho_0) \rho' T$ is the squared Brunt-Vaisala frequency. Under this form, Equation 38 is identical to the entropy part of Andrews’s (1981) APE density and no longer bears any similarity to the integrand of Equation 36.

There appears to be a growing interest in the use of the APE density as a diagnostic tool to obtain insights into the stratified processes. Thus Scotti et al. (2006) extended Holliday & McIntyre’s (1981) framework to allow for an arbitrary reference state and for diabatic effects in the context of the study of internal waves, which was also discussed by Lamb (2007, 2010), with Kang & Fringer (2010) reviewing a number of approaches to defining a local APE density (see also Roulet & Klein 2009 and Molemaker & McWilliams 2010 for studies of turbulent stratified flows).

3. ISSUES IN THE STUDY OF THE GLOBAL ATMOSPHERIC ENERGY CYCLE


The simplest way to describe the atmospheric energy cycle is in terms of the two-component energy cycle given in Section 1. For a steady state, the net APE generation rate $G$ must balance the total viscous dissipation $D$, e.g.,

$$G = \int_V \left( \frac{\epsilon K}{T} \right) \, dq \, dm = D.$$  \hspace{1cm} (39)$$

Equation 39 is reminiscent of the classical Carnot formula for a reversible heat engine, if $D$ is identified with the “useful work.” Ozawa et al. (2003) argued that Lorenz’s APE theory is equivalent to the second law of thermodynamics, as if one approximates $T_R$ as constant and neglects the viscous contribution to $\dot{q}$, then Equation 39 reduces to

$$-T_R \int_V \frac{\dot{q}}{T} \, dm = D$$  \hspace{1cm} (40)$$

because $\int_V \dot{q} \, dm$ is zero in a steady state. Equation 40 is indeed equivalent to the second law because the entropy budget can be written as

$$\int_V \frac{\dot{q}}{T} \, dm + \int_V \frac{\epsilon K}{T} \, dm = 0,$$  \hspace{1cm} (41)$$

provided that $\dot{q}$ excludes viscous heating and that $T_R$ is chosen as the dissipation temperature $T_e$ defined by $1/T_e = \int_V (\epsilon K/T) \, dm/\int_V \epsilon K \, dm$. In this review, however, we insist that Lorenz’s use of a $z$-dependent reference temperature profile $T_R$ is key to his approach and that the use of a constant $T_R$ pertains to exergy theory, not APE. The connection between APE and the second law is important, as it pertains to the debate about whether the atmosphere obeys some extremum principle, with Lorenz speculating that $G$ is maximized, whereas Paltridge (1975) argued that entropy production is maximized (see Lucarini 2009; Pascale et al. 2011, 2012 for review and further discussion).
3.2. The Four-Component Energy Cycle and Waves/Mean Flow Interactions

A key issue in the theory of the large-scale atmospheric circulation is to understand the interactions between the large-scale motions and the strong eddying and wave motions making up most of the weather. Owing to the zonal symmetry of the atmosphere, it has been common to approach the problem by splitting the mean flow into zonal mean and eddy components. Such a decomposition can be applied to Lorenz’s energy cycle, which leads to the four-component energy cycle:

\[
\begin{align*}
\frac{\partial A_Z}{\partial t} &= -C(A_Z, K_Z) - C(A_Z, A_E) + G_Z, \\
\frac{\partial A_E}{\partial t} &= -C(A_E, K_E) + C(A_Z, A_E) + G_E, \\
\frac{\partial K_Z}{\partial t} &= C(A_Z, K_Z) - C(K_Z, K_E) - D_Z, \\
\frac{\partial K_E}{\partial t} &= C(A_E, K_E) - C(K_Z, K_E) - D_E,
\end{align*}
\]

where the subscripts \(Z\) and \(E\) are used to distinguish between a zonal mean and the eddy part, respectively (e.g., Peixoto & Oort 1992). For observational studies of the atmospheric energy cycle, readers are referred to Oort & Peixoto (1974, 1976), Oort (1983), Oort et al. (1989), and Peixoto & Oort (1974).

Following progress in the understanding of wave/mean flow interactions by Andrews & McIntyre (1976, 1978a,b) and Andrews (1983), for instance, Plumb (1983) and Kanzawa (1984) proposed the reformulation of the above four-component energy cycle in terms of the transformed-Eulerian mean theory, which dramatically impacts the form of the various energy conversions, as does the use of different systems of vertical coordinates (e.g., Bleck 1985). Although there have been many attempts at reformulating Lorenz’s four-component energy cycle using various wave/mean flow decompositions and vertical coordinate systems, (e.g., Hayashi 1987, Iwasaki 2001, Uno & Iwasaki 2006, Murakami 2011), these remain complex and technical so that, in practice, the appealing simplicity of Lorenz’s four-component energy cycle makes it the framework of choice to analyze observations (Li et al. 2007) or to assess the performance of climate models (Boer & Lambert 2008).

3.3. The Role of Moisture and Conditional Instability

In the classical description of the atmospheric energy cycle, the atmosphere is usually treated as a dry gas, with moisture assumed to enter the problem only as an additional diabatic term due to latent heat release. From the viewpoint of APE theory, however, this is not satisfactory because the stability properties of dry and moist air are fundamentally different. For dry air, one may easily construct Lorenz’s reference state by sorting the air parcels by ascending potential temperature \(\theta\), which uniquely determines their relative buoyancy. For moist air, however, the relative buoyancies of two moist air parcels with different water mixing ratios depend on the pressure at which they are evaluated. This makes it possible for moist air parcels to be stable for small displacements, but unstable for sufficiently large ones that cause the parcels to condensate and release latent heat, making them more buoyant than their environment. This conditional instability requires that the parcels overcome some energy barrier, called convective inhibition, before they can freely ascend from their level of free convection to their level of neutral buoyancy. The positive work released by a moist air parcel as it goes from its level of free convection to its level of neutral buoyancy
is called the convective available potential energy (CAPE) (Emmanuel 1994, Renno & Ingersoll 1996). Although they are widely used, the concepts of convective inhibition and CAPE are plagued with many conceptual difficulties; because they are based on the so-called parcel method, their values may sensitively depend on the particular moist air parcel chosen to do the computation (de la Torre et al. 2004). More importantly, however, they fail to account for the negative work of buoyancy forces associated with subsiding motions induced by the ascent of buoyant moist air parcels of finite mass.

From the viewpoint of APE theory, the presence of moisture makes it possible for a purely barotropic conditionally unstable atmosphere to possess more than one local potential energy minimum in the space of all possible adiabatic rearrangements of the air parcels. Lorenz (1978, 1979) used the term moist available energy to refer to the vertical component of APE associated with conditional instability. Randall & Wang (1992) developed a similar concept, which they called generalized CAPE, and used it to build a new parameterization of deep cumulus convection (Wang & Randall 1996). Emmanuel (1994) derived an interesting relation between generalized CAPE and CAPE in the case of an atmospheric sounding in which all the boundary-layer air parcels have the same CAPE. Tailleux & Grandpeix (2004) investigated the possibility of defining a generalized convective inhibition and found evidence for some atmospheric soundings of the Southern Great Plains of multiple reference states.

Recently, Pauluis (2007) sought to provide a rigorous generalization of Lorenz’s dry APE framework to a moist atmosphere, by treating moist air as a binary fluid in local thermodynamic equilibrium, allowing for reference states to evolve discontinuously from shallow to deep convective states. Previously, Pauluis & Held (2002a,b) discussed how moist processes could contribute to irreversible entropy production in the atmosphere. By estimating the APE generation associated with idealized atmospheric processes, Pauluis (2007) concluded that the net APE production from processes contributing positively to APE generation appears to greatly exceed all current estimates for the total viscous dissipation. As a result, irreversible APE dissipation processes must exist to make up for the difference. Pauluis’s (2007) framework is an important advance that needs to be further pursued to achieve a better understanding of how the net APE generation rate can be separated into a net APE production and dissipation rate.

4. AVAILABLE ENERGETICS AND THE OCEAN ENERGY CYCLE

4.1. Classical View of the Ocean Energy Cycle

Lorenz’s APE theory was eventually adapted to the study of the oceanic energy cycle by Bryan & Lewis (1979) and others, with the two-component energy cycle usually expressed as follows:

\[ \frac{dK}{dt} = C(A, K) + G_K - D_K, \]  \hspace{1cm} (46)  

\[ \frac{dA}{dt} = -C(A, K) + G_A - D_A, \]  \hspace{1cm} (47)

where \( G_K \) denotes the mechanical power input by the wind, and \( D_K \) is the total viscous dissipation. Roquet & Wunsch (2011) recently reviewed different ways to compute \( G_K \), which is widely estimated to be \( G_K = O(1 \text{ TW}) \). Owing to the importance of the surface in the oceans, the APE generation is commonly decomposed into a net APE production term \( G_A \) by the surface buoyancy fluxes and a net dissipation term \( D_A \) due to turbulent molecular diffusive fluxes of heat and salt. Oort et al. (1989, 1994) estimated \( G_A \approx O(1.2 \pm 0.7 \text{ TW}) \) and concluded that the power inputs due to the wind and surface buoyancy fluxes were comparable.
4.2. Controversy About the Sign of $C(A, K)$

While in the atmosphere, there is no choice but for $C(A, K)$ to be strictly positive to balance $\Delta K > 0$; this is not so in the oceans where $C(A, K)$ in principle can take on both signs. Because one must have $C(A, K) = G_A - D_A$ in steady state, the sign of $C(A, K)$ depends crucially on the magnitude of the APE dissipation. Postulating $D_A$ to be negligible, Peixoto & Oort (1992) assumed $C(A, K) \approx G_A > 0$, but this result conflicts with the results of ocean general circulation models in which $C(A, K)$ is systematically found to be negative if a realistic geometry is used, as first shown by Toggweiler & Samuels (1998) and Gnanadesikan et al. (2005) in ocean-only models and by Gregory & Tailleux (2011) in a range of fully coupled climate models. However, Toggweiler & Samuels (1998) found that $C(A, K)$ is greater than zero in an idealized ocean-sector geometry with no Antarctic Circumpolar Current.

Physically, the reason why $C(A, K)$ can be negative in the oceans is because APE can also be created adiabatically by the wind, in addition to the usual diabatic creation by surface buoyancy fluxes, as illustrated in Figure 4. Because a negative $C(A, K)$ implies a negative APE generation rate $G_A - D_A$, many authors have argued that the oceans should not be regarded as a heat engine. Tailleux (2010) argued, however, that what matters is that the APE production $G_A$ by the surface buoyancy fluxes is large and positive. Indeed, it follows from Figure 4 that a negative $C(A, K)$ implies only that the adiabatic creation of APE by the wind dominates over the diabatic creation by surface buoyancy fluxes, not that the surface buoyancy fluxes are a negligible power source, as is often believed. To illustrate this point, Gregory & Tailleux (2011) suggested diagnosing the local vertically integrated value of $C(A, K)$ as a way to locally discriminate between predominantly wind-driven regions and predominantly buoyancy-driven ones. In a hydrostatic primitive Boussinesq equation model, the vertically integrated APE to kinetic energy conversion is given by $-\int u \cdot \nabla h P dz$, where $u$ and $\nabla h P$ are the horizontal velocity and pressure gradient, respectively, and it is illustrated in Figure 5 for two different coupled climate models. According to this diagnosis, buoyancy-driven regions are found to be primarily linked with western boundary currents, which play a crucial role for the Atlantic meridional overturning circulation (AMOC) (Siip et al. 2012), and are regions traditionally associated with deep-water formation, whereas the primary region of large wind-power input appears to be over the Antarctic Circumpolar Current area, in agreement with theoretical expectations.
4.3. Remaining Challenges and Puzzles

The ocean energy cycle remains plagued with large uncertainties for a number of practical and fundamental reasons. One potentially important source of error stems from the quasi-geostrophic approximation of APE used by Oort et al. (1994), which has long been known in the atmospheric literature (Dutton & Johnson 1967, Taylor 1979) to potentially seriously underestimate both the
APE and APE production rate. Such approximation also neglects the available internal energy component of APE, which Reid et al. (1981) had previously found to be negative and estimated to account only for 10%–20% of the total APE. By using the exact definition of APE, Huang (1998) confirmed that the quasi-geostrophic approximation seriously underestimates APE in the oceans but found that the available internal energy may actually account for up to 40% of the total APE.

In exact APE theory, however, it is important to recognize that, because of the binary character of seawater, the relative densities of the fluid parcels depend on the particular pressure at which they are evaluated. Several reference states may therefore exist, which are each associated with a local potential energy minimum and are separated from each other by some energy barrier. Huang’s (2005) computation assumes that APE must be defined for the reference state achieving the absolute potential energy minimum, but it may be more physical to choose a reference state closer to the actual state, as the actual state may not possess sufficient excess energy in practice to overcome one or several of the above-mentioned energy barriers. These ideas remain poorly understood, however, and their consequences for our understanding of the APE production and dissipation rates remain to be elucidated.

5. APE AND IRREVERSIBLE MIXING IN TURBULENT STRATIFIED FLUIDS

5.1. Evolution of the Reference State and Irreversible Mixing

Because, by construction, the reference state can be affected only by diabatic effects, Winters et al. (1995) suggested that one could monitor its temporal evolution to rigorously quantify irreversible diffusive mixing in turbulent stratified fluids. Thus, for a Boussinesq fluid with a density that obeys the simple diffusive law \( D_\rho = \kappa \nabla^2 \rho \), with \( \kappa \) the molecular diffusivity, Winters et al. (1995) showed that the background potential energy \( G_{\text{PER}} \) in a closed domain evolves in time according to

\[
\frac{dG_{\text{PER}}}{dt} = \int_S g z_R \nabla \rho \cdot \mathbf{n} \, dS - \int_V \kappa \left| \frac{\partial \rho}{\partial z_R} \right|^2 \, dV
\]

(48)

where \( z_R = z_R(\rho, t) \) is the position that a parcel of density \( \rho \) would occupy in the reference state. Equation 48 states that \( G_{\text{PER}} \) is affected by diabatic effects owing to surface buoyancy fluxes \( G_{\text{PER}, \text{buoyancy}} \) and irreversible turbulent mixing \( G_{\text{PER}, \text{mixing}} \). The form of \( G_{\text{PER}, \text{mixing}} \) is interesting because it provides a rigorous definition of the effective turbulent diapycnal diffusivity as

\[
K_\rho = \kappa \left( \frac{||\nabla \rho||^2}{(\partial \rho/\partial z_R)^2} \right) \tag{49}
\]

with \( \langle \cdot \rangle \) an average over constant \( \rho \) surfaces, which is similar to the widely employed Osborn & Cox (1972) model. Equations 48 and 49 provide a far more superior approach to infer \( K_\rho \) in direct numerical simulations of stratified turbulence than that based on inferring it from the highly noisy density flux \( \rho' \mathbf{w}' \) (Caulfield & Peltier 2000, Staquet 2000, Staquet & Bouruet-Aubertot 2001, Peltier & Caulfield 2003). Physically, the temporal evolution of \( \rho_R \) is closely related to that of its probability density function, as discussed by Tseng & Ferziger (2001), which can be used to design algorithms to estimate the reference state that are faster than sorting.

The knowledge of the temporal evolution of \( G_{\text{PER}, \text{R}} \) determines that of \( APE = GPE - G_{\text{PER}, \text{R}} \). It is of interest to derive the local form of its evolution by taking the time derivative of Holliday &
McIntyre’s (1981) APE density (Equation 37):

\[
\frac{DE_{\rho}}{Dt} = (\rho - \rho_R)gw + \nabla \cdot [\kappa g(z - z_R)\nabla \rho] - \rho_0\varepsilon_P + N.L.T.,
\]

where

\[
\rho_0\varepsilon_P = -\kappa g ||\nabla \rho||^2 \frac{\partial \rho}{\partial z} + \kappa g \frac{\partial \rho}{\partial z}.
\]

Equation 51 defines the local APE dissipation rate \(\varepsilon_P\), which is consistent with the form traditionally used to compute the dissipation ratio \(\varepsilon_P/\varepsilon_K\) in stratified turbulence (Oakey 1982) and the mixing efficiency. It is important to point out that the exact local APE evolution equation also possesses an additional nonlocal term (Equation 52) controlled by turbulent molecular diffusion that is lacking in traditional local energetics. Scotti et al. (2006) and Molemaker & McWilliams (2010) suggest that this term may sometimes be positive and larger than \(\rho_0\varepsilon_P\), resulting in turbulent mixing acting locally as a net source of APE. This demonstrates the importance of using the local budget of APE density to identify all possible local sources/sinks of APE, as some of these are filtered out by traditional global APE approaches. Tailleux (2009, 2012a) discussed a possible extension of Winters et al. (1995) to a fully compressible stratified fluid, but further work is needed to extend the framework to a binary fluid, for instance.

### 5.2. Energetics of Mechanically Stirred Horizontal Convection

Although the oceans are nonlinear, oceanographers have nevertheless historically sought to rationalize the large-scale ocean circulation as the superposition of a predominantly horizontal wind-driven circulation and of a buoyancy-driven AMOC. The classical view on the AMOC is that high-latitude cooling and its concomitant deep-water formation must be counteracted by the downward diffusion of heat by turbulent diapycnal mixing if a steady state is to be achieved. However, because turbulent mixing is presumably driven principally by the mechanical stirring due to the wind and tides in the oceans, Munk & Wunsch (1998) and Huang (1999), among others, went as far as challenging the validity of regarding the AMOC as buoyancy driven. This idea subsequently received wide support, as it appeared to agree with a widespread interpretation of Sandström’s (1908) theorem that the surface buoyancy fluxes should be regarded as a negligible source of power and hence of mechanical stirring in the oceans, even though this contradicts Oort et al.’s (1994) conclusion that the buoyancy power input is comparable with that of the wind. As a result, a hypothesis was formulated that the AMOC and its associated meridional heat transport would be negligible in the absence of wind and tides. Laboratory experiments by Whitehead & Wang (2008) provided experimental support for the idea that mechanical stirring could indeed significantly enhance buoyancy-driven horizontal convection. For a general review on horizontal convection, readers are referred to Hughes & Griffiths (2008).

The above ideas renewed interest in studies of ocean energetics (Wunsch & Ferrari 2004, Kuhlbrodt et al. 2007), which all disregarded \(G_A\) as the relevant measure of the buoyancy power input; instead, the focus shifted to the net APE generation rate \(G_A = D_A\) following Paparella & Young (2002) (although this was not initially recognized), which Wang & Huang (2005) estimated to be \(O(15 \text{ GW})\), a very small number compared to \(G_A\). Tailleux (2010) argued, however, that it is no more valid to regard \(G_A = D_A\) as the buoyancy power input as it is to regard \(G_K = D_K\) as the wind-power input. Hughes et al. (2009) and Tailleux (2009) both insisted that \(G_A\) is the relevant measure of the buoyancy power input in the oceans and that it is incorrect to infer from
Sandström’s (1908) study that the surface buoyancy fluxes are a negligible source of power. We note also that Coman et al. (2006) could not reproduce Sandström’s (1908) results.

Tailleux & Rouleau (2010) departed from previous studies by pointing out that APE theory naturally accounts for the possibility of a mechanically controlled buoyancy-driven AMOC, given that the value of $G_A$ depends on the reference state, and hence on any physical processes, such as wind- and tidal-driven turbulent diapycnal mixing, affecting the oceanic stratification. This idea was further made explicit in the context of a wind- and buoyancy-driven Boussinesq ocean with a linear equation of state, through the formula

$$G_A = \frac{\gamma}{1 - \gamma} G_K + \frac{W_{r,\text{laminar}}}{1 - \gamma},$$

(53)

which demonstrates the interdependence of the mechanical power inputs by the wind and buoyancy forcing. In Equation 53, $\gamma = D_d/[D_d + D_K]$ is the bulk mixing efficiency of the oceans, which measures the relative importance of the diffusive dissipation of APE versus the total dissipation, whereas $W_{r,\text{laminar}}$ can be regarded as the net APE generation rate, which Wang & Huang (2005) estimated to be $W_{r,\text{laminar}} = O(15 \text{ W})$.

Tailleux & Rouleau (2010) argued that, owing to the importance of the wind and tides, the last term on the right-hand side of Equation 53 is negligible, allowing the formula to be inverted to predict that

$$\gamma \approx \frac{G_A}{G_A + G_K}.$$  

With $G_A \approx 0.5 \text{ TW}$, and $G_K \approx 1 \text{ TW}$, $\gamma$ is approximately equal to 0.33, which is compatible with estimates of the mixing efficiency estimated from microstructure measurements (e.g., Osborn 1980, Oakey 1982). In the absence of mechanical forcing, the formula predicts that

$$G_A = \frac{W_{r,\text{laminar}}}{1 - \gamma}.$$  

(54)

If $\gamma$ could be regarded as a universal constant close to 20%, as seems to be often assumed in oceanography, Equation 54 would strongly support the idea that $G_A$ would be very small in the absence of the wind and tides. Theoretical and numerical studies reveal, however, that $\gamma \rightarrow 1$ as the Rayleigh number $R_a \rightarrow +\infty$ (Tailleux 2009, Scotti & White 2011). Thus, even though $W_{r,\text{laminar}}$ may be very small, $G_A$ may still be comparatively much larger as $\gamma \rightarrow 1$, suggesting the possibility that strong overturning and heat transport exist, even in the absence of wind and tides.

**SUMMARY POINTS**

1. Lorenz’s/Margules’s globally defined APE theory admits a local formulation in terms of APE density/extended exergy, which represents the natural generalization of the classical thermodynamics theory of available energy to a turbulent stratified fluid.

2. Available energy in stratified fluids appears fundamentally related to the convexity of internal energy in the space of its natural variables. APE measures the distance to a reference-state minimizing potential energy at constant entropy, whereas exergy measures the distance to a reference state in thermodynamic and mechanical equilibrium.

3. Different approaches to available energetics exist, which mainly differ in the way they separate the part of enthalpy available for conversion into kinetic energy from the generation term for available energy. In the oceans, the surface buoyancy production term predicted by exergy theory is about two orders of magnitude larger than that predicted by APE theory, the latter appearing to be the most realistic.
4. The interpretation of energetics and energy conversions in turbulent stratified fluids depends sensitively on the nature of the system of coordinates employed, on whether the Boussinesq approximation is used, and on the particular approach used to separate the mean and eddy fields.

5. The APE budget can be used to provide a rigorous quantification of irreversible turbulent mixing effects in stratified fluids.

6. Binary and multicomponent fluids may admit multiple reference states associated with local potential energy minima and separated from each other by energy barriers, complicating the use of APE theory for such fluids.

7. In the atmosphere, APE production terms greatly exceed all known estimates of viscous dissipation, requiring large APE dissipation mechanisms believed to be associated with moist processes.

8. In the oceans, wind and tidal forcing may control APE production by surface buoyancy fluxes through their control of turbulent mixing and hence of the reference state, whereas buoyancy forcing may partly control the wind-power input by altering surface ocean velocities.

FUTURE ISSUES

1. Further work is needed to fully understand how the APE generation rate splits into a net production and dissipation term in a moist atmosphere.

2. More generally, further work is needed to fully understand how to rigorously apply APE theory to binary and multicomponent fluids. How do we compute the different possible reference states? How do we compute the energy barriers separating them, and the separation between production and dissipation terms?

3. A more systematic investigation and evaluation of the different frameworks for available energetics would be useful to evaluate their predictive power in various typical circumstances.

4. Further work is needed to extend Winters et al.’s (1995) framework to compressible binary and multicomponent fluids, which would potentially help our understanding of double diffusive processes in the atmosphere and oceans.

5. Further work is needed to fully understand how to physically decompose the mean and eddy parts and to achieve an unambiguous physical interpretation of the different energy conversions in terms of actual physical processes.

6. Although there has been some progress in extending APE theory to include nonresting states (e.g., Van Mieghem 1956, Codoban & Shepherd 2003, Andrews 2006, Codoban & Shepherd 2006), further work appears to be needed to fully understand the general principles to deal with arbitrary background mean flows.

DISCLOSURE STATEMENT

The author is not aware of any biases that might be perceived as affecting the objectivity of this review.
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