On the energetics of stratified turbulent mixing, irreversible thermodynamics, Boussinesq models, and the ocean heat engine controversy

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According to current ideas, turbulent mixing accounts for 20 percent or so of the total irreversible kinetic energy (KE) dissipation, by converting KE into mean gravitational potential energy (GPE). In this paper, this idea is refuted by showing that it conflicts both with the second law of thermodynamics, as well as with the result that mean GPE variations averaged over a turbulent mixing event can be of either sign when nonlinearities in the equation of state are accounted for. To understand what happens, the energetics of the fully compressible Navier-Stokes equations is investigated by partitioning both the GPE and in the internal energy (IE) into two and three distinct components respectively. The main results obtained are that: 1) turbulent mixing actually degrades KE into what we call the dead part of the internal energy $IE_0$, exactly like viscous dissipation; 2) the mean GPE variations occur by exchange with what we call the exergy part of internal energy $IE_{exergy}$, exactly as in the laminar case. The reason why the KE dissipated by turbulent mixing cannot end up into mean GPE is because the irreversible $IE_0$-to-$IE_{exergy}$ conversion is extremely inefficient for any nearly incompressible fluid. These simple results have direct and profound implications for the way we think about the forced/dissipated mechanical energy and mean GPE balances in the oceans, which offer a simple solution to a number of riddles about the driving mechanisms of the ocean meridional overturning circulation that have baffled oceanographers for the past century.

1. Introduction

1.1. Viscous versus diffusive dissipation of kinetic energy, gravitational potential energy changes, and mixing efficiency

According to the conventional view of stratified turbulence, turbulent mixing is responsible for a significant fraction — about 20 percent, often called the mixing efficiency — of the total irreversible dissipation of kinetic energy, with the remaining part being achieved by viscous dissipation. As is well known, viscous dissipation degrades KE into internal energy, but in the current view, turbulent mixing is thought in contrast to convert KE into mean gravitational potential energy (GPE). As far as we can judge, the physical basis underlying the current understanding of the energetics of turbulent mixing in stratified fluids has so far relied on the analysis of the L-Boussinesq model

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described in Appendix A, which is based on the incompressible Navier-Stokes equations in the context of the Boussinesq approximation, under the assumption that density can be approximated as a linear function of temperature. The nature of turbulent mixing is most easily understood when GPE is portioned into its available (called AGPE) and non-available (called $GPE_r$) parts in the sense of Lorenz (1955), as originally proposed by Winters & al (1995) and recently reviewed by Peltier & Caulfield (2003). In that case, the energetics of turbulent mixing is neatly summarised into the three following evolution equations for the volume-integrated KE, AGPE, and $GPE_r$:

$$\frac{d(KE)}{dt} = -C(KE, AGPE) - D_{viscous}$$  \hspace{1cm} (1.1)

$$\frac{d(AGPE)}{dt} = C(KE, AGPE) - (W_r - B)$$  \hspace{1cm} (1.2)

$$\frac{d(GPE_r)}{dt} = W_r.$$  \hspace{1cm} (1.3)

As will be further discussed later, it can be shown that the equations as written above apply to both the fully compressible Navier-Stokes equations, as well as the kind of Boussinesq equations given in Appendix A, an important consideration to understand the possible qualitative and quantitative differences between the Boussinesq and Non-Boussinesq equations. In general, however, the particular form of the energy conversions appearing in the above equations critically depends on the set of equations considered.

In the case of the L-Boussinesq model given in Appendix A — which the reader is referred to for the details in notations — and considered by most theoretical and numerical studies of stratified turbulence, e.g. Winters & al (1995); Caulfield & Peltier (2000); Staquet (2000); Peltier & Caulfield (2003), the energy conversion terms appearing in the above equations take on the special form detailed below. First, we have:

$$C(KE, AGPE) = \int_V g \rho w \, dV$$  \hspace{1cm} (1.4)

and is usually called the density flux (denoted by $\Phi_z$ Winters & al (1995)). As clearly explained by Peltier & Caulfield (2003), the density flux is composed of an irreversible component superimposed with a reversible component that usually dominates its instantaneous values. The irreversible component is the one that is of interest for the purpose of parameterising turbulent mixing, e.g. Osborn (1980); Gregg (1987). Second, the viscous dissipation

$$D_{viscous} = \int_V \mu \left( \|\nabla u\|^2 + \|\nabla v\|^2 + \|\nabla w\|^2 \right)$$  \hspace{1cm} (1.5)

and represents the rate of degradation of KE into internal energy (denoted by $\varepsilon$ by Winters & al (1995)). Next, the term $B$ (denoted $\Phi_1$ by Winters & al (1995)) is given by

$$B = \int_V g z \frac{D\rho}{Dt} = \kappa g (\langle \rho \rangle_{bottom} - \langle \rho \rangle_{top})$$  \hspace{1cm} (1.6)

This term represents the Boussinesq approximation of the work of expansion/contraction, and is usually interpreted as representing the background laminar conversion of internal energy into mean GPE that would occur in absence of any macroscopic motion. Finally, the last term

$$W_r = - \int_V g \kappa \|\nabla \rho_r\|^2 \frac{\partial \rho_r}{\partial z_r} = \int_V \kappa \|\nabla \rho_r\|^2 \rho_r N_r^2 \, dV > 0$$  \hspace{1cm} (1.7)
In Winters & al (1995), $W_r$ is denoted by $\Phi_d$, and is written differently in terms of the gradient of $\nabla \rho$, rather than in terms of $\nabla z_r$. We prefer the present formula, as $||\nabla z_r||$ has the simple geometrical interpretation of representing the local increase in surface area of the turbulent isopycnal surface over its laminar value, as discussed in Winters & d’Asaro (1996) and Nakamura (1996) in relation with the concept of effective diffusivity. For reasons that will appear clearly in the remainder of the manuscript, it is useful to collect the terms $W_r$ and $B$ into the following single quantity $D_{\text{diffusive}} = W_r - B$, which can be approximated by

$$D_{\text{diffusive}} \approx -\int_V \kappa \left[ ||\nabla z_r||^2 - 1 \right] \frac{\partial \rho_r}{\partial z_r} dV = \int_V \kappa \left[ ||\nabla z_r||^2 - 1 \right] \rho_r N_r^2 dV > 0 \quad (1.8)$$

provided that $\rho_r = \rho$ near the bottom and surface of the domain considered. The particular notation for $D_{\text{diffusive}}$ is chosen for the obvious reason that it only involves terms that depend on molecular diffusion.

The advantages of partitioning the gravitational potential energy into its available and un-available parts, as originally proposed by Winters & al (1995), can be easily understood by averaging the above equations over a turbulent mixing event, defined here as an episode of intense mixing preceded and followed by laminar conditions for which $AGPE \approx 0$. In that case, denoting the time integral by an overbar, we obtain the following budget equations:

$$\Delta KE = -\overline{C(KE, AGPE)} - \overline{D_{\text{viscous}}} \quad (1.9)$$

$$\Delta AGPE = \overline{C(KE, AGPE)} - \overline{D_{\text{diffusive}}} \approx 0 \quad (1.10)$$

$$\Delta GPE_r = \overline{W_r} \quad (1.11)$$

By combining the first and second equations, one arrives at the result that

$$\Delta KE = -\overline{D_{\text{diffusive}}} - \overline{D_{\text{viscous}}} \quad (1.12)$$

The last equation shows that both molecular diffusion and viscosity yields an irreversible dissipation of kinetic energy. As mentioned above, the relative importance of the non-viscous and viscous dissipations is traditionally measured by the concept of mixing efficiency, which we define here by:

$$\gamma_{\text{mixing}} = \frac{D_{\text{diffusive}}}{D_{\text{viscous}} + D_{\text{diffusive}}} \quad (1.13)$$

following the definition proposed by Peltier & Caulfield (2003).

In the above model, internal energy is not explicitly accounted for, but there is a consensus to say that viscous dissipation converts KE into internal energy. In contrast, however, the widespread idea is that the diffusive dissipation of KE via $D_{\text{diffusive}}$ is associated with the irreversible conversion of $KE$ into $GPE_r$ via the route $KE \rightarrow AGPE \rightarrow GPE_r$. In fact, this idea leads Peltier & Caulfield (2003) to suggest that the budget of $GPE_r$ should be written down as:

$$\Delta GPE_r = D_{\text{diffusive}} + B \quad (1.14)$$

by using the fact that $D_{\text{diffusive}} = W_r - B$. Peltier & Caulfield (2003) argue that this form is preferable to reflect the fact that the increase in $GPE_r$ is the sum of a contribution from turbulent mixing that converts $KE$ into $GPE_r$, and from laminar diffusion via $B$ which converts internal energy into $GPE_r$. 
1.2. Criticism of the standard view

In the standard view, the physical mechanism whereby KE is irreversibly dissipated into $GPE_r$ is thought to result first from the reversible adiabatic conversion of KE into $AGPE$, followed by the irreversible conversion of $AGPE$ into $GPE_r$ at the rate $D_{\text{diffusive}} = W_r - B$, as illustrated schematically in Fig. 1. Meanwhile, internal energy also contributes to the $GPE_r$ increase at the rate $B$, so that the total rate at which $GPE_r$ increases is $W_r = D_{\text{diffusive}} + B$. Although such a view is not unpalatable at first sight, thermodynamicists would probably be concerned about its reliance on the use of an irreversible mechanical-to-mechanical energy conversion — the $AGPE \rightarrow GPE_r$ conversion — which, as far as we can judge, is unheard of in thermodynamics. Indeed, much more familiar in the present context are irreversible mechanical-to-internal and internal-to-mechanical energy conversions. Since perfectly efficient irreversible mechanical-to-mechanical conversions are not documented, it would seem prudent to examine first whether the assumed irreversible $AGPE \rightarrow GPE_r$ conversion would not actually stand as a shortcut for an indirect conversion of the form $AGPE \rightarrow IE \rightarrow GPE_r$, which has the advantage of involving only well-known energy conversions. Only if such a possibility can be discarded then does it become reasonable to seek a patent for the idea of a perfectly efficient $AGPE \rightarrow GPE_r$ direct conversion. If one adopts the indirect scenario, however, one must immediately abandon the idea that $GPE_r$ increases at the expenses of $AGPE$, because then the second law of thermodynamics tells us that only a tiny fraction of the $AGPE$ being converted into $IE$ can actually be converted into $GPE_r$. Does that mean that the indirect route $AGPE \rightarrow IE \rightarrow GPE_r$ is physically inconsistent?

In order to clarify this point, let us try to identify what could have originally been construed as a reasonable physical basis for assuming the existence of a direct irreversible route $AGPE \rightarrow GPE_r$. As far as we can judge, this assumption seems to be based on the fact that $W_r$ and $D_{\text{diffusive}}$ are nearly identical and positive in the L-Boussinesq model. This property, however, is easily shown to be an artifact of assuming a linear equation of state in the L-Boussinesq model. Indeed, we show in Appendix A that if a nonlinear equation of state is assumed, then the expression for $B$ and $W_r$ become:

$$B = \kappa g \langle \rho \rangle_{\text{bottom}} - \langle \rho \rangle_{\text{top}} - \int V \kappa \rho g \frac{d\rho}{dT} \|\nabla T\|^2 dV,$$

$$W_r = \int V \rho g \kappa \|\nabla z_r\| \frac{\partial (\alpha_r z_r)}{\partial z_r} \frac{\partial T_r}{\partial z_r} dV.$$

According to Eq. (1.15), $B$ is no longer expected to be small and positive, but instead large and negative, whereas Eq. (1.16) shows that $W_r$ can actually become negative when $d(\alpha z)/dz < 0$. In fact, the latter case plays a central role in Fofono (1998, 2001) which show that it is easily encountered in the oceans. For this reason, this case will be called the Fofono regime thereafter, to distinguish it from the case described by the L-Boussinesq model, which we will call the classical regime. While both $W_r$ and $D_{\text{diffusive}}$ appear to be critically affected the nonlinearity of the equation of state, it is interesting to note that in contrast, the difference $D_{\text{diffusive}} = W_r - B$ appears to be hardly unchanged compared to the L-Boussinesq model, owing to the cancellation of the terms due to the nonlinearity of the equation of state, as discussed in Appendix A. What all this show therefore is that in general, there is no reason to expect $W_r$ and $D_{\text{diffusive}}$ to be similar, as they can even have opposite signs!

Since the common view that $W_r$ and $D_{\text{diffusive}}$ are nearly identical is easily refuted, we therefore see no particular strong reason to hold on to the idea of the existence of the direct irreversible $AGPE \rightarrow GPE_r$ route, so that the present paper is about val-
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1.3. Consequences for the forced/dissipated mechanical energy balance in the oceans

If one accepts for a moment the idea that turbulent mixing actually degrades kinetic energy into internal energy, i.e., exactly as viscous dissipation, and not into \( GPE_r \), then it should be immediately clear that this must have profound implications for the way we think about the forced/dissipated mechanical energy in stratified fluids. For instance, the mechanical energy balance in the oceans is usually written as follows:

\[
W_{\text{mech}} + \int_V P \frac{DV}{Dt} dV = D_{\text{viscous}}
\]  

(1.17)

where \( W_{\text{mech}} \) is the work of the wind stress say, \( B \) is the work of expansion/contraction, and \( D_{\text{viscous}} \) is the viscous dissipation. Because neither \( W_{\text{mech}} \) and \( D_{\text{viscous}} \) can’t possibly be identified as the source of mechanical energy supplied by the thermodynamic forcing (surface heat fluxes say), this role has generally been attributed to \( B \). What is puzzling about \( B \), however, is that it does not reveal any explicit dependence on the integrated cooling/heating \( Q_{\text{cooling}} \) and \( Q_{\text{heating}} \), in contrast with \( W_{\text{mech}} \), which in the case of an ocean forced by the wind stress shows an explicit dependence upon the latter, viz:

\[
W_{\text{mech}} = \int_S \tau \cdot u_s dS.
\]  

(1.18)

As a result, the issue of whether surface heating and cooling actually provides significant mechanical work to the oceans has been an issue of debate for the past century, starting with the work of Sandström (1908). Over the past decade, the problem has been investigated by numerical ocean models whose mechanical energy balance is similar to Eq. (1.17), where \( B \) is replaced by its Boussinesq approximation \( B_{OOGCM} \). It was first shown by Toggweiler & Samuels (1998) that if \( B_{OOGCM} \) may happen to be positive for particular idealised ocean geometries, such as a sector ocean model, it is usually negative for a realistic geometry, a result since confirmed by independent studies in ocean-only models, e.g. Gnanadesikan & al. (2005), as well as in fully coupled climate models (J. Gregory, 2007, personal communication). This led some authors to propose to regard the oceans as a heat engine only in the case where \( B > 0 \), and as a mechanically-driven system otherwise, e.g. Nycander & al. (2007). This result has done much to reject the idea that the work of buoyancy fluxes is anything significant in the oceans, Wang & Huang (2005);
Figure 1. Idealised depiction of the diffusive route for kinetic energy dissipation. (I) represents the laminar state possessing initially no AGPE and AIE, but some amount of KE. (II) represents the state obtained by the reversible adiabatic conversion of some kinetic energy into APE, which increases APE but leaves the background \( GPE_r \) and \( IE_r \) unchanged; (III) represents the state obtained by letting the horizontal part of molecular diffusion smoothes out the isothermal surfaces until all the APE in (II) has been converted in background \( PE_r = GPE + IE_r \).

Wunsch & Ferrari (2004). Some authors go as far as saying that the oceans should be viewed as having a negative thermodynamic efficiency e.g. Huang (2004).

Now, the idea according to which we should regard the oceans as mechanically-driven whenever \( B < 0 \) seems to overlook the fact that in that case, \( B \) appears to be associated with a net KE to IE conversion, a transformation usually associated with dissipation, and hence with the idea of spin-down rather than spin-up. In any case, the idea that the reverse of \( B > 0 \) implies a mechanically-driven system is certainly not straightforward. In fact, if we accept the idea that turbulent mixing represents an additional dissipation term, then one is naturally tempted to rewrite the mechanical energy balance (1.17) as follows:

\[
W_{\text{mech}} + (B + D_{\text{diffusive}}) = D_{\text{viscous}} + D_{\text{diffusive}},
\]

and wonder whether it is the term \( W_{\text{buoyancy}} = B + D_{\text{diffusive}} \) that should be regarded as the work of buoyancy fluxes rather than \( B \)? As it turns out, this seems to be the case, as discussed in details in Section 4. The latter section also addresses the forced/dissipated budget of background \( GPE_r \), which will revisit Munk & Wunsch (1998)'s theory according to which turbulent mixing balances the loss of GPE due to cooling at high-latitudes. Finally, section 5 summarizes and discusses the main results.

2. A new look at the energetics of freely decaying turbulence

2.1. Boussinesq versus Non-Boussinesq energetics

The main approach followed in this paper to clarify the energetics of turbulent mixing was to derive a new formulation of the energetics of the fully compressible Navier-Stokes equations based on the classical partitioning of the GPE and IE into their available and un-available (or background) components, as defined by Lorenz (1955), as well as a new (in the present context) partitioning of the background IE into a dead and
exergy parts, $IE_0$ and $IE_{\text{exergy}}$ respectively, taking inspiration in the classical work of Gibbs (1878). Physically, $IE_0$ represents the internal energy of an isothermal state having exactly the same total energy as Lorenz (1955)’s reference state. In other words, $IE_0$ is the internal energy that Lorenz (1955)’s reference state would eventually achieve after total relaxation toward thermodynamic equilibrium. The exergy component of the IE is then simply defined as $IE_{\text{exergy}} = IE_r - IE_0$. Our definition is similar in spirit with other definitions of exergy used in engineering literature, e.g. Bejan, A. (1997). In contrast with Lorenz (1955)’s available energy, which is a form of energy available only through adiabatic and reversible transformations, exergy is a form of energy that is only available through irreversible transformations. It is therefore complimentary to Lorenz (1955)’s APE. With regard to the available part of IE, it is simply defined as $AIE = IE - IE_r$. The reader interested in the technical aspects of the derivations will find them in Appendix B. By comparing the energetics of fully compressible Navier-Stokes equations with that of the L-Boussinesq and NL-Boussinesq models given in Appendix A, we arrived at the conclusion that the energetics of these different models can be written in the following common format:

$$\frac{d(KE)}{dt} = -C(KE, APE) - D_{\text{viscous}} \quad (2.1)$$

$$\frac{d(APE)}{dt} = C(KE, APE) - D_{\text{diffusive}} \quad (2.2)$$

$$\frac{d(GPE_r)}{dt} = W_r \quad (2.3)$$

where the explicit expressions for $C(KE, APE)$ and $D_{\text{diffusive}}$ depend on the particular set of equations considered, as summarised in Tab. 1. Note that the fully compressible NSE also possess a term related to the work done by the atmospheric pressure for a
system with a free surface. This term is found to be usually small (see also McDougall (2003)), and is therefore systematically neglected in this paper for simplicity and clarity.

While the above equations are in essence similar to those of Winters et al (1995) or Peltier & Caulfield (2003), one of the main results of this paper is the derivation of the following evolution equations for the dead and exergy components of the internal energy:

$$\frac{d(IE_0)}{dt} = (1 - \gamma_0)(D_{viscous} + D_{diffusive}) \approx D_{viscous} + D_{diffusive}$$ (2.4)

$$\frac{d(IE_{exergy})}{dt} = \gamma_0(D_{viscous} + D_{diffusive}) - W_r \approx -W_r, \quad (2.5)$$

where the important parameter $\gamma_0$ acts as a thermodynamic efficiency which scales as:
Figure 4. (a) The increase of AGPE as a function of the stirring energy SE (see text for details). Each point represents a different stratification shuffled by a different random permutation. The continuous line represents the straight-line of equation $\Delta AGPE = SE$ which would describe the energetics of the stirring process if $AIE$ were zero, i.e., if the fluid were perfectly incompressible; (b) the change of $GPE_r$ as a function of the stirring energy $SE$ dissipated by diffusive mixing; the dotted line is the straight line of equation $\Delta GPE_r = \text{Diffusive dissipated } SE$ which would describe the energetics of turbulent mixing if the irreversible conversion $AGPE \rightarrow GPE_r$ existed; (c) The change in dead internal energy $IE_0$ as a function of the diffusively dissipated stirring energy $SE$. The dashed line is the straight-line of equation $\Delta IE_0 = \text{diffusively dissipated } SE$; the figure shows a near perfect correlation; (d) The change in $GPE_r$ as a function of the exergy change. The dashed line is the straight line of equation $\Delta GPE_r = -\Delta IE_{exergy}$. The figure shows again a near perfect correlation.

$$\psi_0 = O \left( \frac{\alpha P}{\rho C_p} \right)$$

(2.6)

for some values of $\alpha$, $P$, $\rho$ and $C_p$ typical of the domain considered. The important point is that $\psi_0$ is very small for nearly incompressible fluids. For seawater, for instance, typical values encountered in laboratory experiments done at atmospheric pressure are $\alpha = 2.10^{-4} \text{K}^{-1}$, $P = 10^5 \text{Pa}$, $C_p = 4.10^3 \text{J.K}^{-1}.\text{kg}^{-1}$, $\rho = 10^3 \text{m}^3.\text{kg}^{-1}$, which yield $\alpha P/\rho C_p = 5.10^{-6}$.

In the deep oceans, this value can increase up to $O(10^{-3})$, but this is still very small. Eqs. (2.4) and (2.5) confirm the idea that the APE dissipated by turbulent mixing ends up in the dead part of internal energy $IE_0$, as the viscously-dissipated $KE$, after which the second law imposes that only a tiny fraction $\psi_0$ can pursue its way to $IE_{exergy}$ and then into the background $GPE_r$. As a result, the term $D_{\text{diffusive}}$ is best interpreted as being distinct as $W_r$, and the fact that these two terms
appear nearly identical in the L-Boussinesq model must be recognised as an artificial property of assuming a linear equation of state which ceases to be satisfied for more general conditions.

The fact that the energetics of the Boussinesq and Non-Boussinesq can be written down under a common format is important, because it suggests that the qualitative characteristics of the turbulent mixing are not crucially dependent on the incompressibility assumption, and in particular on the assumed smallness of the work of expansion/contraction $B$. According to Table 1, $D_{\text{diffusive}}$ appears to have a similar form in the Boussinesq and Non-Boussinesq equations, which suggests that the mixing efficiency $\gamma_{\text{mixing}} = D_{\text{diffusive}}/(D_{\text{diffusive}} + D_{\text{viscous}})$ may be accurately simulated even possibly in the L-Boussinesq model. This conclusion is also supported by the result derived in Appendix A that although both $W_r$ and $B$ are strongly altered by the nonlinearities of the equation of state, this is not so for their difference $W_r - B$. A possible source of worry, however, concerns the difference between the Boussinesq and Non-Boussinesq equations regarding the KE/APE conversion, which is given by $C(KE, APE) = W$ for the former, but by $C(KE, APE) = W - B$ in the latter. This would be of no concern if $B$ was indeed much smaller than $W$, as is the case in the L-Boussinesq model, but as discussed above, and further confirmed below, $B$ should be expected to be large and negative in realistic circumstances. If the energy equations are averaged over a turbulent mixing event, defined here as an episode of intense mixing preceded and followed by quasi-laminar conditions for which $APE \approx 0$, then one sees that

$$\overline{C(KE, APE)} \approx D_{\text{diffusive}}$$

where the overbar denotes a time average. This is important, because it suggests that the cancellation between $W_r$ and $B$ due to the nonlinearities also applies to $W - B$. Note that this concerns only the irreversible part of $C(KE, APE)$; this is not expected to be a problem, though, because the reversible part of $B$ is expected to be very small. While one may be reasonably confident that the mixing efficiency can be simulated with good accuracy regardless of the model considered, this is not so with regard to the variations of $GPE_r$ which crucially depend on whether the nonlinearities equation of state are considered or not. If one is interested in simulated $GPE_r$ accurately, then clearly the L-Boussinesq model is not sufficient, and the NL-Boussinesq model appears to be needed.

2.2. A simple experimental test of the new versus old theory

As is often the case when presented with a new theory that conflicts with commonly accepted ideas, a natural question, asked by several colleagues, is whether its predictions can be tested experimentally. Although the question is legitimate, it is not entirely fair, because the prediction of the standard theory that turbulent mixing converts $AGPE$ irreversibly into $GPE_r$ has never been tested experimentally either, despite having been around for much longer. Instead, the never-verified hypothesis that turbulent mixing converts $AGPE$ irreversibly into $GPE_r$ appears to have been the basis for experimental determination of the mixing efficiency, under the assumption that measuring the rate of increase of $GPE_r$ approximates well the rate at which turbulent mixing dissipates $AGPE$, which itself relies on the result that $D_{\text{diffusive}} \approx W_r$ in the L-Boussinesq model. As argued above, however, there is no theoretical basis to expect that $W_r$ and $D_{\text{diffusive}}$ should be similar in the most general circumstances, greatly complicating the task of measuring mixing efficiencies experimentally if we accept the idea that $GPE_r$ variations are no longer a good proxy for $D_{\text{diffusive}}$.

While we are still unclear about how the predictions of each theory could be tested by means of laboratory experiments, it is rather easy, on the other hand, to propose
a simple idealised experimental protocol that can clearly discriminate between the two old and new theory of turbulent mixing. To that end, the reader is asked to visualise a piece of thermally stratified fluid initially lying in Lorenz (1955)'s reference state in a two-dimensional container with a flat bottom, vertical walls, and a free surface exposed at constant atmospheric pressure at its top. The fluid is then discretised on a rectangular array so that each discrete fluid element has the same mass $\Delta m = \rho \Delta x \Delta z$, where $x$ and $z$ are the horizontal and vertical coordinates respectively, as illustrate in Fig. 3. In order to obtain reliable statistics, both the stratification and the mean pressure were randomly selected.

For each randomly generated stratification, all discrete fluid elements are then numbered from 1 and $N$ and then randomly shuffled by generating a random permutation of $N$ elements, such that each parcel conserves its entropy (or equivalently its potential temperature). Stirring the parcels in such way requires an external amount of energy — called the stirring energy $SE$ — which is entirely determined by the permutation generated. Since we know the temperature and pressure of the shuffled state, it is possible to diagnose both the internal energy and gravitational potential energy, assuming the pressure to be hydrostatic, so that the stirring energy $SE$ is diagnosed from:

$$SE = (GPE + IE)_{shuffled} - (GPE + IE)_{initial}$$  \hspace{1cm} (2.8)

The random shuffling is intended to mimic the stirring phase occurring in real turbulent fluid associated with the adiabatic $KE \rightarrow APE$ energy conversion. Because the transformation is adiabatic, the background quantity are not affected, so that energetically, what happens is:

$$\Delta APE = \Delta AGPE + \Delta AIE = SE$$  \hspace{1cm} (2.9)

$$\Delta GPE_r = \Delta IE_r = 0$$  \hspace{1cm} (2.10)

Thousands of stratification were generated. The results of the experiments are depicted in Fig. 4, which plots the $\Delta AGPE$ as a function of $SE$.

In a second step, all the fluid parcels having the same pressure are set to the same in-situ temperature such that their total enthalpy is conserved. Such a process is intended to mimic the destruction of APE by the horizontal part of molecular diffusion. From an energetic viewpoint, the process transforms APE into background $PE_r$ according to:

$$\Delta APE = \Delta AGPE + \Delta AIE = -qSE$$  \hspace{1cm} (2.11)

$$\Delta GPE_r + \Delta IE_r = qSE$$  \hspace{1cm} (2.12)

where $q$ is a parameter close to unity. Although the mixing process describes makes the temperature a function of depth alone, there is no guarantee that the resulting stratification be stable. $q = 1$ only when the stratification is statically stable. Panel (b) depicts the increase in background $GPE_r$ as a function of the dissipated stirring energy $qSE$. The dashed line represents $GPE_r = qSE$, which corresponds to the prediction of the L-Boussinesq model. The figure shows that although there are a number of instances where the predictions of the L-Boussinesq model are satisfied, $GPE_r$ actually decreases for the majority of the experiments considered here, which corresponds to the Fofonoff regime discussed above. Panel (c) and (d) depict respectively the variation of dead internal energy $IE_0$ as a function of the dissipated stirring energy $qSE$, and the change in background $GPE_r$ as a function of the change in exergy $IE_{exergy} = IE_r - IE_0$. In both cases, the correlation is visually near perfect, which strongly supports the relations:

$$\Delta IE_0 \approx qSE$$  \hspace{1cm} (2.13)
Figure 5. (Top panels) The work of expansion/contraction $B$ normalised by its laminar value as a function of a normalised $APE$ for a particular stratification corresponding to the classical regime, with the right panel being a blow-up of the left panel. (Bottom panels) Same as above figure, for the same temperature stratification, but taken at a mean pressure of 50 dbar instead of atmospheric pressure, which is sufficient to put the system in the Fofonoff regime. The figures show that although $B$ is usually negative in every case, it is nevertheless positive for small values of $APE$ in the classical regime, as expected from L-Boussinesq theory.

$$\Delta GPE_r \approx -\Delta (IE_r - IE_0)$$

(2.14)

Thus, the experimental protocol clearly supports the new hypothesis proposed in this paper, and refutes the standard assumption that it is possible to dissipate $AGPE$ into $GPE_r$. 

2.3. Numerical estimates of $B$, $W_r$ and $D_{\text{diffusive}}$ as a function of $APE$

In order to conclude this section, the above results are further illustrated by providing numerical estimates of the important conversion terms $B$, $W_r$ and $D_{\text{diffusive}}$ which we will depict as a function of the $APE$ present in the system. We first discuss the case of the expansion/contraction $B$, which we suggested should be large and negative when the nonlinearities of the equation of state are retained, based on the computation given in Appendix A. To that end, note that if $\rho$ is regarded as a function of temperature and pressure, then $B$ can be written exactly as follows:

$$B = \int_V \frac{\alpha P}{\rho C_p} \nabla \cdot (\kappa \rho C_p \nabla T) \, dV + \int_V \frac{\alpha P}{\rho C_p} \rho c_s^2 \, dV - \int_V \frac{P}{\rho c_s^2} \frac{DP}{Dt} \, dV.$$  

(2.15)

By comparison of this expression with the corresponding Boussinesq approximations of it, it is clear that the Boussinesq approximations are directly comparable only with the first term in this expression. In order to get a sense for $B$, we computed this term from the same methodology as in Section 2, by computing the heating function from a
Figure 6. (Left panels) The rate of change of $GPE_r$ normalised by its laminar value as a function of normalised $APE$, in the classical regime (top panel) as well as in the Fofono regime (bottom panel). The stratification is identical to that of Fig. 5. (Right panels) The rate of diffusive dissipation of $APE$ normalised by $W_r$ laminar value, as a function of a normalised $APE$, in the classical regime (top panel), as well as for the Fofono regime (bottom panel). The figure illustrates the fact that if former can be regarded as a good proxy for the latter in the classical regime, as is usually assumed, this is clearly not the case in the Fofono regime. The two figures also illustrate the fact that the former always underestimate the latter for a thermally stratified fluid, so that observed values of mixing-efficiencies obtained from measuring $GPE_r$ variations are necessarily lower-bounds for actual mixing efficiencies.

In the non-Boussinesq equations, $W_r$ is given by:

$$W_r = \int_V \frac{\alpha_r P_r}{\rho_r C_{pr}} \nabla \cdot (\kappa \rho C_p \nabla T) \, dV + \int_V \frac{\alpha_r P_r}{\rho_r C_{pr}} \left( 1 - \frac{T_r}{T} \right) \{ \nabla \cdot (\kappa \rho C_p \nabla T) + \rho \varepsilon \} \, dV.$$  

(2.16)

In this expression, only the first integral is directly comparable with the Boussinesq analogue derived in Appendix A. Owing to the presence of the multiplicative factor $(1 - T_r/T)$, the second integral is expected to be much smaller than the first integral.
The first integral term in Eq. (2.16) was computed as a function of $APE$ for the same two particular stratifications used in Fig. 5. The result is depicted in the left panels of Fig. (6), with the top and bottom panels corresponding to the classical and Fofono regimes respectively. As in the Boussinesq equations, $W_r$ is positive in the classical regime, and negative in the Fofono regime.

In Section 2, we also suggested that the strong correlation between the rate of change of $GPE_r$ and the diffusive dissipation rate of $KE$ that is expected from the classical L-Boussinesq model only occurs for a linear equation of state, but that such correlation should possibly be weak otherwise. To test this, we also computed $D_{\text{diffusive}} = -H$ as a function of $APE$ for the same two particular stratifications, the results being depicted in the right panels of Fig. 6, with the top and bottom panels corresponding to the classical and Fofono regimes respectively. The function $H$ was computed using the first integral in its exact formula:

$$H = \int_V \left( \frac{T - T_r}{T} \right) \nabla \cdot (\kappa \rho C_p \nabla T) \, dV + \int_V \left( \frac{T - T_r}{T} \right) \rho \varepsilon \, dV \quad (2.17)$$

In that case, we find that $-H$ is very similar in both regimes, similarly as for the NL-Boussinesq equations discussed in Section 2 and in appendix A. We experimentally verified that $-H$ is a good approximation for $W_r - B$, not shown. This figure therefore confirms the distinct character of the diffusive dissipation and that of the $GPE_r$ variations. Their common link, however, is to behave as either increasing or decreasing function of $APE$, depending on whether the classical or Fofono regimes is achieved.

2.4. Synthesis

A synthesis of the above results is depicted in the energy diagram Fig. 7. The top two panels illustrate the classical and Fofono regimes. The bottom panel shows when the $AGPE$ and $AIE$ are subsumed into $APE$, the resulting energy diagram is basically identical to that of the Boussinesq equation as per to the new interpretation proposed in the bottom panels of Fig. 2. According to the two top panels, the energetics of turbulent mixing in the Fofono regime appears to differ in fundamental ways to that in the extensively studied regime. Most notably, whereas both $W$ and $B$ act as net sinks of $KE$ in freely decaying turbulence, this no longer appears to be the case in the Fofono regime. Indeed, in that case, a fraction of the $KE$ dissipated into $AIE$ appears to be able to be recycled back into $KE$. In that case, Fofono (1998, 2001) speculated that the decrease of $GPE_r$ would act as a positive feedback on turbulent kinetic energy, thereby enhancing mixing until getting back to the classical regime. In our opinion, Fofono’s proposed mechanism cannot occur in reality, because it would require an irreversible mechanical-to-mechanical conversion of $GPE_r$ into $AGPE$, which we criticise for the same reasons that we criticise the standard view promoting the reverse conversion in the first place. As a result, there seems to be no other choice for $GPE_r$ but to be converted into $IE_{\text{exergy}}$, which supports the conclusions of McDougall (2003), although not for the reasons invoked by the latter study. What happens exactly in the Fofono’s regime is therefore a totally open question, whose mysterious nature warrants further investigation.

3. Forced/dissipated balances in the oceans

3.1. A new approach to the mechanical energy balance in the oceans

In some sense, the finding that turbulent mixing degrades kinetic energy into dead internal energy, exactly like viscous dissipation, and not into mean $GPE_r$ as previously
thought, amounts to the discovery of a new dissipation mechanism, with profound implications for the way we think about forced/dissipated balances in the stratified fluids, as illustrated below in the case of the oceans. Indeed, as long as turbulent mixing was thought to convert KE into GPE, it was an internal conversion term in the total mechanical energy balance, in no way inconsistent with the idea that mechanical energy dissipation is ultimately achieved by molecular viscosity at the Kolmogorov’s scale, e.g. Thorpe (2005). The latter idea has to be completely revisited, however, if one accepts the idea that turbulent mixing is also able to degrade KE into internal energy, as then viscous dissipation at the Kolmogorov’s scale is no longer the only possible way to dissipate KE irreversibly.

Let us illustrate the implications of the above idea in the case of a thermally stratified ocean forced mechanically by wind forcing and thermodynamically by surface heating and cooling. We are not sure yet of how to deal with the impact of salinity, on which we hope to report in a subsequent study. Based on the above ideas, we would like to write

---

**Figure 7.** The energetics of freely decaying turbulence for the classical regime (top panel), the Fofonoff regime (middle panel), and a synthesis of both regimes obtained by subsuming AGPE and AIE into APE alone. Note the similarity of the energetics in the lower panel and that of the re-interpreted Boussinesq energetics of the lower panels of Fig. 2.
down the mechanical energy balance under the following form:

$$W_{\text{mech}} + W_{\text{buoyancy}} = D_{\text{diffusive}} + D_{\text{viscous}},$$ (3.1)

with $W_{\text{buoyancy}}$ hopefully turning out to be a positive number quantifying the work of surface heat fluxes. In Appendix B, we showed that the evolution equations for KE and APE in a forced/dissipated stratified fluid are given by:

$$\frac{d(KE)}{dt} = -C(KE, APE) - D_{\text{viscous}} + W_{\text{mech}}$$ (3.2)

$$\frac{d(APE)}{dt} = C(KE, APE) + H,$$ (3.3)

so that for a steady state, the mechanical energy balance becomes:

$$\overline{W}_{\text{mech}} + \overline{H} = D_{\text{viscous}},$$ (3.4)

where the overbar denotes a time average over a sufficiently long period that a statistical balance can be assumed to hold, where $H$ is defined by (see Eq. (B21):

$$H = \int_V \rho \left(1 - \frac{T_r}{T}\right) \dot{Q} dV = \int_V \left(\frac{T - T_r}{T}\right) \nabla (\kappa \rho C_p \nabla T) + \rho \varepsilon \right) dV.$$ (3.5)

As seen previously, we expect the part of $H$ depending on molecular diffusivity to be strictly negative in freely decaying turbulence, i.e., in absence of external forcing, whereas the sign of the part related to viscous dissipation can probably be of any sign, depending on the spatial distribution of $\rho \varepsilon$. In a forced system, however, one needs to isolate the contribution due to the surface heat fluxes. To that end, an integration by parts is used to rewrite $H$ as follows:

$$H = \int_S \left(\frac{T - T_r}{T}\right) \kappa \rho C_p \nabla T \cdot \mathbf{n} dS - \int_V \kappa \rho C_p \nabla T \cdot \nabla \left(\frac{T - T_r}{T}\right) dV + \int_V \left(\frac{T - T_r}{T}\right) \rho \varepsilon dV$$

$$= \gamma^+ C_{\text{heating}} - \gamma^- C_{\text{cooling}} - D_{\text{diffusive}} + \gamma \varepsilon D_{\text{viscous}},$$ (3.6)
where $n$ is a unit normal vector to the domain, while we also defined:

$$D_{\text{diffusive}} = \int_V \kappa \rho C_p \nabla T \cdot \nabla \left( \frac{T_r - T}{T} \right) dV$$  \hspace{1cm} (3.7)$$

$$\gamma_e D_{\text{viscous}} = \int_V \left( \frac{T - T_r}{T} \right) \rho e dV$$  \hspace{1cm} (3.8)$$

$$\gamma_c^+ = \frac{\int_{S^+} (T - T_r) QdS}{\int_{S^+} QdS}, \quad \gamma_c^- = \frac{\int_{S^-} (T - T_r) QdS}{\int_{S^-} QdS}$$  \hspace{1cm} (3.9)$$

where $S^+$ and $S^-$ are the fraction of the surface where the surface heating $Q = \kappa \rho C_p \nabla T$. $n$ is positive and negative respectively, such that

$$\int_{S^+} QdS = Q_{\text{heating}}, \quad \int_{S^-} |Q|dS = Q_{\text{cooling}}.$$  \hspace{1cm} (3.10)$$

Note that $Q_{\text{heating}}$ and $Q_{\text{cooling}}$ are positive by construction. In the above formula, the coefficients $\gamma_c^+, \gamma_c^-$, and $\gamma_e$ are naturally interpreted as thermodynamic efficiencies, owing to their similarity with a classical Carnot efficiency, although we cannot always guarantee that their sign should be positive. Note also that although $D_{\text{diffusive}}$ is expected to be strictly positive based on the previous considerations, as well as from the numerical illustrations presented in the previous section, its assumed positiveness is not explicitly apparent in Eq. (3.7). Although there may exists an unambiguous mathematical proof establishing the positive definite character of Eq. (3.7), as was shown to be possible in the L-Boussinesq and NL-Boussinesq models in Appendix A, we haven’t yet succeeded in finding it in the fully compressible case. For this reason, we can only posit the positive definiteness character of the diffusive part of $H$ as a conjecture. In any case, inserting the above expression for $H$ into Eq. (3.4) yields:

$$\mathbf{W}_{\text{mech}} + \gamma_c^+ \mathbf{Q}_{\text{heating}} + \gamma_c^- \mathbf{Q}_{\text{cooling}} = \mathbf{D}_{\text{diffusive}} + (1 - \gamma_e) \mathbf{D}_{\text{viscous}}.$$  \hspace{1cm} (3.11)$$

In the oceans, it is usually agreed that $W_{\text{mech}} = O(1 \text{ TW})$ e.g. , Wunsch & Ferrari (2004), while $D_{\text{diffusive}}$ is expected to account for about 20 percent of the viscous dissipation term, e.g. Peltier & Caulfield (2003). Therefore, only the terms depending on $Q_{\text{heating}}$ and $Q_{\text{cooling}}$ need to be discussed, which requires estimating the thermodynamic efficiencies $\gamma_c^+$ and $\gamma_c^-$. To that end, let us point out that the magnitude of these efficiencies depend on the discrepancy between the parcels’s actual temperature and that they would have in the reference state. In the oceans, because the regions of net heating are usually the warmest, it follows that the reference temperature of the parcels must be close to their actual temperature, which leads us to regard $\gamma_c^+$ as a negligible number. Because of the bowl-shaped structure of the temperature distribution in the oceans, it seems clear that the reference temperature of the surface parcels differ more and more as one goes poleward. Obviously, one must have $T_r > T$ in the regions of cooling. The difference between $T_r$ and $T$ is determined by the adiabatic lapse rate, for which a typical value is $\Gamma = O(4 \text{ mK}/\text{MPa})$ for a salinity of 35 PSU between the surface and about 1000 dbar, e.g. Feistel (2003), so that for a reference pressure of about 1000 dbar = 10 MPa, then $T - T_r = O(40 \text{ mK})$, and $1 - T/T_r = O(10^{-4})$ for $T = 273 \text{ K}$. Our value for the thermodynamic efficiency $\gamma_c^- = O(10^{-4})$ is about three orders of magnitude larger than the one derived by Wang & Huang (2005) obtained by not recognising the need to remove the effect of turbulent mixing from the work of expansion/contraction. If we then take $Q_{\text{cooling}} = O(2 \text{ PW})$, we obtain $\gamma_c^- Q_{\text{cooling}} = O(0.2 \text{ TW})$, which is smaller, but comparable, with the work done by the wind stress. This result therefore contrasts with the
conclusions of Wunsch & Ferrari (2004) that our current best estimate for the work of surface buoyancy fluxes is zero. This result is important because it naturally explains why cooling, rather than heating, is the dominant contribution of surface buoyancy fluxes to the production of mechanical energy in the oceans. This result clearly supports the idea that deep water formation is the main driver of the meridional overturning circulation, as was the standard assumption until about a decade ago when it started to be criticised by Munk & Wunsch (1998)’s theory, as well as from the numerical results from Toggweiler & Samuels (1998). As far as we can judge, the present results resolve the longstanding controversy originating from Sandstrom (1908)’s paper of whether surface buoyancy fluxes can significantly contribute to the production of mechanical energy in the oceans. The answer here is clearly positive, although further work is needed to obtain a physically more rigorous number, which is beyond the scope of this exploratory paper.

3.2. A new look at the GPE\(_r\) balance and Munk & Wunsch (1998)’s theory

Similarly, we can look at the GPE\(_r\) balance, whose evolution equation is given by:

\[
\frac{d(GPE_r)}{dt} = W_r, \tag{3.12}
\]

and which for a steady state simply reduces to \(W_r = 0\). Using the explicit formula for \(W_r\) derived in Appendix B yields:

\[
W_r = \int_V \frac{P_r \alpha_r T_r}{\rho_r C_{pr} T} \left[ \nabla \cdot (\kappa \rho C_p \nabla T) + \rho \varepsilon \right] dV = 0. \tag{3.13}
\]

As in the previous section, the idea is to separate the “free” behaviour of \(W_r\) as occurs in freely decaying turbulence, from the forced part that results from the surface heating and cooling, by using an integration by parts, which yields:

\[
\overline{W}_r = \int_V \frac{P_r \alpha_r T_r}{\rho_r C_{pr} T} (\rho \varepsilon) dV + \int_S \frac{P_r \alpha_r T_r}{\rho_r C_{pr} T} \rho \kappa C_p \nabla T \cdot \mathbf{n} dS - \int_V \kappa \rho C_p \nabla T \cdot \nabla \left( \frac{P_r \alpha_r T_r}{\rho_r C_{pr} T} \right) dV
\]

\[= \Upsilon_\varepsilon \overline{D} + \Upsilon^+ \overline{Q}_{heating} - \Upsilon^- \overline{Q}_{cooling} + \overline{W}_{r,mixing} = 0, \tag{3.14}\]

where \(\Upsilon_{heating}\) and \(\Upsilon_{cooling}\) have the same meaning as in the previous paragraph, \(\overline{W}_{r,mixing}\) is the change of GPE\(_r\) resulting from turbulent mixing, defined by:

\[
\overline{W}_{r,mixing} = -\int_V \kappa \rho C_p \nabla T \cdot \nabla \left( \frac{P_r \alpha_r T_r}{\rho_r C_{pr} T} \right) dV \tag{3.15}
\]

while \(\Upsilon_\varepsilon\), \(\Upsilon^+\) and \(\Upsilon^-\) are suitably defined values of \(\Upsilon\),

\[
\Upsilon_\varepsilon = \frac{\int_V \frac{P_r \alpha_r T_r}{\rho_r C_{pr} T} (\rho \varepsilon) dV}{\int_V \rho \varepsilon dV}, \quad \Upsilon^+ = \frac{\int_{S+} \frac{P_r \alpha_r T_r}{\rho_r C_{pr} T} Q dS}{\int_{S+} Q dS}, \quad \Upsilon^- = \frac{\int_{S-} \frac{P_r \alpha_r T_r}{\rho_r C_{pr} T} Q dS}{\int_{S-} Q dS} \tag{3.16}
\]

where \(S+\) and \(S-\) have the same meaning as before. Eq. (3.14) can be regarded as the rigorous counterpart of Munk & Wunsch (1998), obtained here deductively from first principles, without using any turbulent closure. Physically, it expresses the fact that the mean GPE\(_r\) balance is achieved between: 1) a term related to the conversion of the viscous Joule heating into mean GPE\(_r\); 2) the conversion of surface heating into mean GPE\(_r\); 3) the cooling effect; 4) turbulent mixing. In their paper, Munk & Wunsch (1998) neglected the first two effects to retain only the last two. To fix ideas, it is useful to try to come up with order of magnitude estimates for each term. Using \(\Upsilon_\varepsilon = 10^{-3}\), and \(\overline{D} = 10^{12}\) W, the first term is \(O(1\ GW)\). For the surface heating term, it is convenient to
use $Y^+ = 5.10^{-6}$, $Q_{\text{heating}} = 10^{15}$ W, so that the second term is $O(5$ GW$)$. Regarding to the cooling, $Y^- = 5.10^{-4}$, $Q_{\text{cooling}} = 10^{15}$ W, so that the third term is $O(0.5$ TW$)$. Based on the previous numbers, the $GPE_r$ balance appears to be well approximated by the following balance:

$$y^- Q_{\text{cooling}} \approx W_{r,\text{mixing}},$$

(3.17)

where the loss due to cooling is balanced by turbulent mixing, which is exactly Munk & Wunsch (1998)'s point. On this account, the present results establish that there is a rigorous theoretical basis for the latter study. Nevertheless, there are important points in which the present analysis strongly differs from that of Munk & Wunsch (1998). The first point pertains to the interpretation of the turbulent mixing, which is interpreted as a conversion term between turbulent kinetic energy and GPE in MW98, where our study established that the conversion is actually between the exergy part of the internal energy and the background $GPE_r$. The second point is that our expression for turbulent mixing is significantly different from that of MW98 which relies on the classical expression:

$$\frac{d(GPE)}{dt}_{\text{mixing}} = \int V K_{\rho} \rho N^2 dV.$$

(3.18)

We need to point out that the latter expression is only a very crude approximation to the exact term appearing in Eq. (3.17). Such an approximation can be recovered from the exact expression if one makes the approximation $\nabla T \approx \nabla T_r$ and $T \approx T_r$ within the integral, while also neglecting the vertical variations of $\alpha_r/(\rho_r C_{pr})$, in which case one may write:

$$\int V \kappa C_p \nabla z_r \| \nabla z_r \|^2 \frac{\partial T_r}{\partial z_r} \frac{\partial}{\partial z_r} \left( \frac{P_r}{\rho_r C_{pr}} \right) dV \approx \int V \kappa g \| \nabla z_r \|^2 \frac{\partial \rho_r}{\partial z_r} dV \approx \int V K_{\rho} \frac{\partial \rho_r}{\partial z_r} dV$$

(3.19)

which is similar to Eq. (3.18) if one uses the definition of turbulent eddy diffusivity $K_{\rho}$ proposed by Winters & al (1995), and also neglects the compressibility effect due to pressure so that one can write: $-(g/\rho) \partial \rho_r/\partial z_r = N^2$. In the oceans, however, none of these approximations are particularly good, so that the numerical estimates of MW98 should be treated with extreme caution.

4. Conclusions

In this paper, we questioned the physical basis for the widespread idea that the fraction $\gamma_{\text{mixing}}$ of the KE irreversibly dissipated by turbulent mixing ends up into mean GPE. Instead, what our analysis suggests is that turbulent mixing degrades kinetic energy into “dead” internal energy, exactly like viscous dissipation, while the variations of mean GPE result from an exergy exchange with the exergy part of internal energy, exactly as in the laminar case. To exist, the irreversible conversion of KE into mean GPE would require a perfectly efficient conversion between the dead and exergy components of the internal energy, but this is forbidden by the second law of thermodynamics. Apparently, the present confusion stems from the fact that in the L-Boussinesq equations — which has remained so far the main theoretical framework to discuss the energetics of stratified turbulence — the rate $D_{\text{diffusive}}$ at which turbulent mixing dissipates KE is nearly equal to the rate $W_r$ of increase of mean GPE, which misled fluid dynamicists in thinking that the two processes should be coupled. When the nonlinearities of the equation of state are accounted for, however, $D_{\text{diffusive}}$ and $W_r$ can possibly become very different and even opposite in sign, which clearly conflicts with the idea that the two processes are directly related. Nonlinearities of the equation of state also cause the work of expansion/contraction to
be negative and as large as the other conversion terms in a turbulent fluid, and not second order as is commonly thought. Technically speaking, this implies that stratified should be regarded as compressible, even for fluid flows at low Mach numbers, and not incompressible as is usually thought. Fortunately, these potentially large compressibility effects do not appear to play a crucial role for the understanding of turbulent mixing, for they appear to cancel out in the expression for $D_{\text{diffusive}}$, which in turns implies that they must also cancel out on average in the conversion between kinetic energy and available potential energy. Therefore, as long as the main nonlinearities of the equation of state are retained, there does not appear to be major issues in using the incompressible Boussinesq equations to simulate turbulent mixing. The use of a linear equation of state, on the other hand, appears to be valid only over a limited range of circumstances, as it cannot describe, for instance, the physically important Fofonoff regime for which $GPE_r$ decreases as the result of turbulent mixing. With regard to the important concept of mixing efficiency, our results show that of the two definitions: 1) the mixing efficiency measures the relative importance of the non-viscous dissipation of KE over the total irreversible dissipation of KE; 2) the mixing efficiency is the fraction of the kinetic energy that is irreversibly converted into mean GPE, only definition 1) is correct, as it is clear from our results that definition 2) does not make sense. In this regard, it is found that $GPE_r$ variations always underestimate the amount of KE dissipated by turbulent mixing in thermally stratified fluids, which implies that any published value of mixing efficiency obtained by measuring the evolution of $GPE$ in laboratory experiments is expected to be a lower bound for actual value of mixing efficiency.

The present results are important, because they basically amount to the discovery of a "new" dissipation mechanism in stratified fluids. Indeed, even if turbulent mixing has been known for decades to be partly responsible for the irreversible decay of KE, it has never really been regarded as a proper dissipation mechanism before, this role being usually attributed exclusively to viscous dissipation. If one agrees, however, that turbulent mixing should be regarded as an actual dissipation mechanism — that is, one degrading kinetic energy into internal energy, as viscous dissipation — then one must also realise that this must have profound implications for the way we think about the forced/dissipated mechanical energy (i.e., $KE + APE$) and non-available GPE budgets in stratified fluids. When applied to the particular case of the oceans, this idea appears to be able to naturally resolve a number of existing controversies and riddles that have baffled oceanographers for the past century, and presently the cause of enormous confusion in the field, as illustrated by Zickfeld & al. (2007). Among other things,

(a) It resolves the longstanding controversy originating with the work of Sandström (1908)'s about whether surface heat fluxes can produce any significant amount of work in the oceans, with the attached question of whether it is legitimate to regard the ocean meridional overturning circulation as being driven by deep water formation resulting from cooling at high-latitudes. In this paper, the issue is clarified by showing that the work of expansion/contraction $B$, which is the only term directly affected by surface heat fluxes, should be written as $B = W_{\text{buoyancy}} - D_{\text{diffusive}}$, where $D_{\text{diffusive}}$ is the fraction of the mechanical energy dissipated by turbulent mixing, and $W_{\text{buoyancy}}$ the work of the surface heat fluxes. This decomposition shows that $B$ can be small or even negative without this implying that the work of the surface heat fluxes is necessarily small and negative, because it is erroneous to identify the two. According to this decomposition, it does not appear to be justified to regard the oceans as a heat engine only when $B > 0$, and as a mechanically-driven otherwise, as suggested by Toggweiler & Samuels (1998); Gnanadesikan & al. (2005); Nycander & al. (2007), because the negative part of $B$ corresponds to dissipation of mechanical energy, as viscous dissipation. To conclude as to
whether the oceans are a heat engine, it is more appropriate to discuss its nature from the
sign of $W_{\text{buoyancy}}$. A rough estimate for actual oceans provides $W_{\text{buoyancy}} = O(0.2 \text{TW})$, which is not only positive, but also not much smaller than current estimates of about $O(1 \text{TW})$ for the work of the wind stress, e.g. Wunsch & Ferrari (2004).

(b) It confirms Munk & Wunsch (1998)’s theory that turbulent mixing balances the loss of $GPE_r$ due to cooling from first principles, without resorting to the approximate advective/diffusion balance or any other turbulent closure. We point out, however, that Munk & Wunsch (1998)’s arguments only apply to the background $GPE_r$ budget, that is, the part of GPE unavailable for conversion into KE, and therefore have no bearing whatsoever on understanding the driving mechanisms of the ocean meridional overturning circulation, as these can only be addressed from the view point of available energetics as is well understood since Lorenz (1955)’s seminal paper. In particular, it is incorrect to regard Munk & Wunsch (1998)’s theory as implying that the oceans are not a heat engine, or that turbulent mixing drives the ocean overturning circulation. This is because turbulent mixing and cooling are respectively a dissipation and source term from the viewpoint of available energetics. Furthermore, Munk & Wunsch (1998) interpreted their results as implying that the overturning circulation is mechanically-driven on the assumption that the increase of $GPE_r$ due to turbulent mixing is achieved at the expenses of the mechanical energy input. Our results show, however, that $GPE_r$ increase at the expenses of the exergy part of internal energy, which itself is fed by surface heat fluxes. On the other hand, while our results refute the idea of a direct mechanical forcing of the $GPE_r$ increase, it must be pointed out, however, that the mechanical energy input is crucial in controlling the rate of conversion between $IE_{\text{exergy}}$ and $GPE_r$, so that in some sense, it is still possible to speak of a mechanical-control of the overturning circulation.

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Appendix A. Energetics of Incompressible Navier-Stokes Equations

A.1. Boussinesq equations with equation of state nonlinear in temperature

The purpose of this appendix is to document the energetics of the Boussinesq system of

equations that form the basis for most inferences about stratified turbulence for fluid flows

at low Mach numbers, and which is commonly used in the theoretical and numerical study

do turbulence, e.g. Winters & al (1995); Caulfield & Peltier (2000); Staquet (2000); Peltier & Caulfield (2003). However, because such a model usually assumes a linear

equation of state in temperature, it lacks sufficient generality for the present purposes,

so that the model actually considered also allows for the possibility of an equation of

state nonlinear in temperature. The resulting set of equations is therefore as follows:

$$\frac{Dv}{Dt} + \frac{1}{\rho_0} \nabla P = -\frac{g \theta}{\rho_0} z + \mu \nabla^2 v \quad (A 1)$$
\[ \nabla \cdot \mathbf{v} = 0 \]  
(A 2)

\[ \frac{DT}{Dt} = \kappa \nabla^2 T \]  
(A 3)

\[ \rho(T) = \rho_0 \left[ 1 - \int_{T_0}^T \alpha(T')dT' \right] \]  
(A 4)

where \( \mathbf{v} = (u, v, w) \) is the three-dimensional velocity field, \( P \) is the pressure, \( \rho \) the density, \( T \) the temperature, \( \mu \) the viscosity, \( \kappa \) the molecular diffusion, \( g \) the acceleration of gravity, and \( \rho_0 \) a reference density. The classical Boussinesq model, called L-Boussinesq model in this paper, is simply recovered by taking \( \alpha \) to be a constant in Eq. (A 4). In that case, Eqs. (A 3) and (A 4) may be combined to obtain the following diffusive model for density:

\[ \frac{\partial \rho}{\partial t} = \kappa \nabla^2 \rho \]  
(A 5)

as assumed in many numerical studies of turbulence, e.g. Winters & al (1995); Caulfield & Peltier (2000); Staquet (2000); Peltier & Caulfield (2003). The general model for which \( \alpha \) is taken to increase with temperature will be called the NL-Boussinesq model.

### A.2. Standard energetics

Evolution equations for the KE and GPE are obtained by the standard procedure, e.g. Batchelor (1967); Landau & Lifshitz (1987), under the assumption that the system is isolated. The first equation is premultiplied by \( \rho_0 \mathbf{v} \) and volume-integrated. After re-organisation, the equation can be written as:

\[ \frac{d(KE)}{dt} = \frac{d}{dt} \int_V \frac{\rho_0 v^2}{2} dV = - \int_V \rho_0 g w dV + \int_V \rho_0 \varepsilon dV \]  
(A 6)

where \( W \) is the so-called density flux, \( D \) is the viscous dissipation rate of kinetic energy, and \( W_{mech} \) is the mechanical energy input due to the applied stress. Let us now define the total gravitational potential energy of the fluid as the volume integral of \( \rho g z \), whose time evolution is given by:

\[ \frac{d(GPE)}{dt} = \frac{d}{dt} \int_V \rho g z dV = \int_V \rho g w dV - \int_V \rho_0 g z \kappa \nabla^2 T dV, \]  
(A 7)

where \( B \) is the Boussinesq approximation of the work of expansion/contraction. In the present case, it is possible to derive an explicit analytical formula for \( B \):

\[ B = - \int_V g z \rho_0 \alpha \kappa \nabla^2 T dV = \int_V \kappa g \rho_0 \alpha \frac{\partial T}{\partial z} dV + \int_V \kappa \rho_0 g z \frac{d\alpha}{dT}(T) \| \nabla T \|^2 dV \]

\[ = \kappa g \left[ \langle \rho \rangle_{\text{bottom}} - \langle \rho \rangle_{\text{top}} \right] + \int_V \kappa \rho_0 g z \frac{d\alpha}{dT}(T) \| \nabla T \|^2 dV, \]  
(A 8)

by using integration by parts, and using the fact that the surface term vanishes because the surface is by assumption located at \( z = 0 \), where \( \langle \rho \rangle_{\text{bottom}} \) and \( \langle \rho \rangle_{\text{top}} \) denote the surface-integral of the bottom and top value of density. It is important to remark that only the term \( B_L \) appears when a linear equation of state is used. Such a term will in general be small in turbulent conditions, owing to the smallness of the molecular diffusivity \( \kappa \), as well as to the boundedness of the top-bottom density contrast. In the
present case, the nonlinearity of the equation of state is seen to play a crucial role, as it shows that the second term may a-priori become arbitrarily large as the turbulence increases.

A.3. Lorenz (1955)'s available energetics

We now seek evolution equations for the available and un-available parts of the gravitational potential energy, as previously done by Winters & al (1995) in the case of the L-Boussinesq equations. By definition, the expression for the $GPE_r$ is given by:

$$GPE_r = \int_V \rho_r z_r dV,$$  \hspace{1cm} (A 9)

where $z_r = z_r(x, t)$ and $\rho_r = \rho_r(z_r, t)$ are the vertical position and density of the parcels in the Lorenz (1955)'s reference state. In Boussinesq models, the reference state is usually constructed by having the fluid parcels conserving their potential temperature, here taken equal to their temperature in absence of compressibility effects, so that $T_r(z_r, t) = T(x, t)$. Taking the time derivative of Eq. (A 9) thus yields:

$$\frac{d(GPE_r)}{dt} = \int_V g z_r \frac{D\rho_r}{Dt} dV + \int_V g \rho_r \frac{Dz_r}{Dt} dV = - \int_V g z_r \rho_0 \kappa \nabla^2 T dV$$

$$= \int_V \kappa \rho_0 g \nabla T \cdot \nabla (\alpha z_r) dV = \int_V \kappa \rho_0 g \|\nabla z_r\|^2 \frac{\partial (\alpha z_r)}{\partial z_r} \frac{\partial T_r}{\partial z_r} dV$$  \hspace{1cm} (A 10)

In order to obtain the above formula, we used the result obtained by Winters & al (1995) that the term depending on $\frac{Dz_r}{Dt}$ vanishes in the above integral, and the fact that

$$\nabla[T(x)] = \nabla[T_r(z_r(x))] = \frac{\partial T_r}{\partial z_r} \nabla z_r$$  \hspace{1cm} (A 11)

$$\nabla[\alpha(T) z_r] = \nabla[\alpha(T_r(z_r))] = \frac{\partial (\alpha z_r)}{\partial z_r} \nabla z_r$$  \hspace{1cm} (A 12)

The equation for the available part of $GPE$ then become:

$$\frac{d(AGPE)}{dt} = \frac{d(GPE)}{dt} - \frac{d(GPE_r)}{dt} = W - (W_r - B) = W - D_{\text{diffusive}}$$  \hspace{1cm} (A 13)

where

$$D_{\text{diffusive}} = \int_V \rho_0 g \kappa \left\{ \|\nabla z_r\|^2 \left( \frac{\partial (\alpha z_r)}{\partial z_r} \frac{\partial T_r}{\partial z_r} - \frac{\partial \alpha}{\partial T} \|\nabla T\|^2 \right) \right\} dV - B_L$$

$$= \int_V \rho_0 g \kappa \|\nabla z_r\|^2 \frac{\partial T_r}{\partial z_r} dV - B_L > 0$$  \hspace{1cm} (A 14)

by using the results that:

$$\int_V \rho_0 g \kappa \frac{\partial \alpha}{\partial T} \|\nabla T\|^2 dV = \int_V \rho_0 g \kappa \|\nabla z_r\|^2 \left( \frac{\partial T_r}{\partial z_r} \right)^2 dV$$  \hspace{1cm} (A 15)

$$\int_V \rho_0 g \kappa \|\nabla z_r\|^2 \frac{\partial (\alpha z_r)}{\partial z_r} \frac{\partial T_r}{\partial z_r} dV = \int_V \rho_0 g \kappa \|\nabla z_r\|^2 \left( 1 + \left( \frac{\partial \alpha}{\partial T} \frac{\partial z_r}{\partial T_r} \frac{\partial T_r}{\partial z_r} \right) \frac{\partial T_r}{\partial z_r} \right) dV.$$  \hspace{1cm} (A 16)

Interestingly, we see while both $W_r$ and $B$ are both strongly altered by the nonlinear behaviour of $\alpha$, this is not so of their difference. This will have important consequences for assessing the validity of the Boussinesq approximation, as discussed in Section 2. Moreover, this expression clearly establishes that the diffusive dissipation term $D_{\text{diffusive}}$...
is in general quite different from $W_r$, i.e., the rate of increase/decrease of $GPE_r$, when nonlinearity of the equation of state is important. This is in contrast with what happens in the L-Boussinesq model.

Appendix B. Energetics of compressible Navier-Stokes Equations

B.1. Compressible Navier-Stokes Equations (CNSE)

The purpose of this appendix is to generalise Winters & al (1995)’s results to the fully compressible Navier-Stokes equations, which are written here in the following form:

$$\rho \frac{Dv}{Dt} + \nabla P = -\rho g \hat{z} + \nabla \cdot S \quad (B1)$$

$$\frac{D\rho}{Dt} + \rho \nabla \cdot v = 0 \quad (B2)$$

$$\frac{D\Sigma}{Dt} = \frac{\dot{Q}}{T} = \frac{\rho \varepsilon - \nabla \cdot F_q}{\rho T} \quad (B3)$$

$$I = I(\Sigma, v), \quad P = P(\Sigma, v) = -\frac{\partial I}{\partial v}. \quad (B4)$$

In the present description, the three-dimensional Eulerian velocity field $v = (u, v, w)$, the specific volume $\nu = 1/\rho$ (with $\rho$ the density), and the specific entropy $\Sigma$ are taken as the dependent variables, with the thermodynamic pressure and temperature being diagnostic variables as expressed by Eqs. (B4-B5). Further useful notations are: $D/Dt = \partial/\partial t + (v \cdot \nabla)$ is the substantial derivative, $\varepsilon$ is the dissipation rate of kinetic energy, $F_q = -k_T \rho C_p \nabla T$ is the diffusive heat flux, $C_p$ is the specific heat capacity at constant pressure, $k_T$ is the molecular diffusivity for temperature, $g$ is the acceleration of gravity, $\mu$ is the kinematic viscosity, and $z$ is an normal unit vector pointing upward. Moreover, $S$ is the deviatoric stress tensor:

$$S_{ij} = \mu \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) + \left( \lambda - \frac{2\mu}{3} \right) \delta_{ij} \frac{\partial u_{\ell}}{\partial x_{\ell}} \quad (B6)$$

in classical tensorial notation, e.g., Landau & Lifshitz (1987), where Einstein’s summation convention for repeated indices has been adopted, and where $\delta_{i,j}$ is the Kronecker delta. The parameters $\mu$ and $\lambda$ are the shear and bulk (or volume) viscosity respectively.

B.2. Standard energetics

The derivation of energy evolution equations for the compressible Navier–Stokes equations is a standard exercise, e.g. de Groot & Mazur (1962); Landau & Lifshitz (1987), so that we only give the final results. In the standard description of energy, only the volume-integrated kinetic energy (KE), gravitational potential energy (GPE), and internal energy (IE) are considered, viz.,

$$KE = \int_V \frac{\rho \nu^2}{2} dV, \quad GPE = \int_V \rho g z dV, \quad IE = \int_V \rho I(\Sigma, v) dV, \quad (B7)$$

whose standard evolution equations are respectively given by:

$$\frac{d(KE)}{dt} = -\int_V \rho g w dV + \int_V P \frac{DV}{Dt} dW + W_{mech} - D - P_a \frac{dV_{ol}}{dt}, \quad (B8)$$
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\[ \frac{d(GPE)}{dt} = \int_V \rho gw \, dV, \quad (B\,9) \]

\[ \frac{d(IE)}{dt} = \int_V \rho \dot{Q} \, dV - \int_V \rho \left( \frac{Dv}{Dt} \right) \, dm = D_{\text{viscous}} + Q_{\text{heating}} - Q_{\text{cooling}} - B, \quad (B\,10) \]

where \( W_{\text{mech}} \) is the mechanical energy input, \( Q_{\text{heating}} \) (resp. \( Q_{\text{cooling}} \)) is the integrated value of the heating (resp. cooling) flowing through the boundary of the fluid, and \( V_d \) is the total volume of the fluid; further definitions and justifications are given further down the text. The sum of all the above equations yields an evolution equation for the total energy \( E = KE + GPE + IE \):

\[ \frac{dE}{dt} = W_{\text{mech}} + Q_{\text{heating}} - Q_{\text{cooling}} - P_a \frac{dV_d}{dt}, \quad (B\,11) \]

which expresses the fact that the total energy of the fluid is modified by the mechanical energy input due to the external applied stress, the net values of surface-integrated heating and cooling, and work of the atmospheric pressure on volume changes. As these derivations are quite standard, justifications for the above equations are only briefly outlined. Thus, the KE equation (B 8) is classically obtained by multiplying the momentum equation by \( v \), and integrating over the volume domain. The term \( W \) results from the product of \( v \) with the gravitational force vector, whereas the product \( v \cdot \nabla P = \nabla \cdot (Pv) - P \nabla \cdot v = \nabla \cdot (Pv) - (P/v)Dv/Dt \) yields the work of expansion/contraction minus the work done by the atmospheric pressure against total volume changes. The product of the velocity vector by the stress tensor is written as the sum \( W_{\text{mech}} \), where \( W_{\text{mech}} \) represents the work input due to the external stress, and \( D \) the positive dissipation of kinetic energy. The general expression for the mechanical energy input is:

\[ W_{\text{mech}} = \int_{\partial V} vS \cdot ndS = \int_{\partial V} \tau \cdot vdS \quad (B\,12) \]

where \( vS \) is the vector of component \( (Sv)_j = S_{ij} u_i \), while \( Sn = \tau \) is the stress applied along the surface boundary enclosing the fluid. \( W_{\text{mech}} \) is therefore the work of the applied stress done against the fluid velocity. If one assumes no-slip boundary condition on all solid boundaries, then this work is different from zero only on the free surface. The function \( D \) is the function of dissipation:

\[ D = \int_V \left\{ \mu \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3} \delta_{ij} \frac{\partial u_k}{\partial x_k} \right)^2 + \lambda (\nabla \cdot v)^2 \right\} dV, \quad (B\,13) \]

where again the summation convention for repeated indices has been used, e.g., Landau & Lifshitz (1987). The equation for \( GPE \) (B 9) is simply obtained by taking the time derivative of its definition, using the fact that \( D(\rho g z dV)/Dt = \rho gw \), since \( D(\rho dV)/Dt = 0 \) from mass conservation. The equation for \( IE \) (B 10) results from the fact that the differential of internal energy in the entropy/specific volume representation is given by \( dI = Ta \Sigma - Pdv \). The term \( Q_{\text{heating}} \) and \( Q_{\text{cooling}} \) represents the surface-integrated net heating and cooling respectively going through the surface enclosing the domain.

B.3. Available energetics

In this paragraph, both the GPE and IE are partitioned into an available and un-available (or background) component, as initially proposed by Lorenz (1955), building upon ideas
functions of \( P \).

where the defining integrals can be written either in the actual state or reference state, namely: 1) the total potential energy in an adiabatic re-arrangement of the parcels; 2) the specific entropy \( S \) and mass \( \rho \) of the parcel, viz.,

\[
\Sigma(x, t) = \Sigma(x_r, t) = \Sigma_r(z_r, t),
\]

(\ref{eq:S})

where the second condition can be equivalently formulated in terms of the Jacobian \( J = \partial(x_r)/\partial(x) \) of the mapping between the actual and reference state as follows:

\[
\rho(x, t) = \rho(x_r, t) \frac{\partial(x_r)}{\partial x} = \rho_r(z_r, t) \frac{\partial(x_r)}{\partial \xi}.
\]

(\ref{eq:rho})

The main goal is to derive evolution equations for the GPE and IE of the reference state, which are defined by

\[
GPE_r = \int_V \rho(x, t) g z_r(x, t) dV = \int_{V_r} \rho_r(z_r, t) g z_r dV_r,
\]

(\ref{eq:GPE})

\[
IE_r = \int_V \rho I(\Sigma_r, v_r) dV = \int_{V_r} \rho I(\Sigma_r, v_r) dV_r,
\]

(\ref{eq:IE})

where the defining integrals can be written either in the actual state or reference state, the passage from one space to the other being governed by the classical rules pertaining to the change of coordinates.

Prior to deriving evolution equations for \( GPE_r \) and \( IE_r \), let us mention three important properties of the reference state, namely: 1) \( \rho_r = \rho_r(z_r, t) \) and \( P_r = P_r(z_r, t) \) are functions of \( z_r \) alone (and time); 2) \( \rho_r \) and \( P_r \) satisfy the hydrostatic balance at all times, i.e., \( \partial P_r/\partial z_r = -\rho_r g \) (this follows from the reference state being the state minimising the total potential energy in an adiabatic re-arrangement of the parcels; 3) The velocity \( v_r = (Dx_r/\partial t, Dy_r/\partial t, Dz_r/\partial t) \) of the parcels in the reference state satisfies the usual mass conservation equation:

\[
\frac{Dv_r}{\partial t} = v_r \nabla r \cdot v_r,
\]

(\ref{eq:mass_conservation})

which follows from the properties of the mapping from the actual to the reference state.

While the first two conditions are well known, and previously recognised by Lorenz (1955) and others, the third condition appears to be formulated here for the first time. Interestingly, the latter condition stresses the point that Lorenz (1955)’s reference state — despite the appearances — is not a state of rest, for all the parcels are in motion; moreover, while their density \( \rho_r \) and \( P_r \) depend on \( z_r \) at any given time, their velocity \( v_r \) will in general depend on all three coordinates. Finally, we require the fluid parcels in the reference state to stay within the same physical domain, and therefore to satisfy the no-normal flow condition along solid boundaries, and the free surface condition \( w_r = \partial n_r/\partial t \) at \( z_r = \eta_r(t) \).

We can now compute the evolution equations for \( GPE_r \) and \( IE_r \) by taking the total differential of all the elements with Eqs. (\ref{eq:GPE}) and (\ref{eq:IE}), which, after some manipulation, yields the following expressions:

\[
\frac{d(GPE_r)}{\partial t} = \int_V \rho_r \frac{Dz_r}{\partial t} dV + \int_V g z_r \frac{D(\rho dV)}{\partial t} = \int_V \rho_r g w_r dV = W_r
\]

(\ref{eq:evolution})
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\[
\frac{d(I_{E_r})}{dt} = \int_V \rho \left[ T_r \frac{D\Sigma_r}{Dt} - P_r \frac{Dv_r}{Dt} \right] dV = \int_V \rho \frac{T_r}{T} \hat{Q} dV - P_a \frac{dV_{ol,r}}{dt} - \int_V \rho_r g w_r dV,
\]

\[
= \int_V \rho \hat{Q} dV - \int_V \rho \left( 1 - \frac{T_r}{T} \right) \hat{Q} dV - P_a \frac{dV_{ol,r}}{dt} - W_r,
\]

\[
= -H + \int_V \rho \varepsilon dV - \int_{\partial V} \mathbf{F}_q \cdot \mathbf{n} dS - P_a \frac{dV_{ol,r}}{dt} - W_r,
\]

\[
= -H + D + Q_{heating} - Q_{cooling} - P_a \frac{dV_{ol,r}}{dt} - W_r. \tag{B 21}
\]

Eq. (B 20) results from each parcel conserving their mass, which implies \( D(\rho dV)/Dt = 0 \), and from posing \( w_r = Dz_r/Dt \); Eq. (B 21) uses the equality of entropy in the reference and actual states, which implies \( D\Sigma_r/Dt = D\Sigma_r/Dt = Q/T \); the net surface heating was written as the sum of its positive \( Q_{heating} \) and negative \( Q_{cooling} \) parts. Moreover, Eq. (B 19) was also used to derive the following expression:

\[
\int_V \rho P_r \frac{Dv_r}{Dt} dV = \int_V \rho P_r \frac{Dv_r}{Dt} dV_r = \int_V P_r \nabla_r \cdot \mathbf{v}_r dV_r = \int_{\partial V_r} \mathbf{P}_r \mathbf{v}_r \cdot \mathbf{n}_r dS_r - \int_V \mathbf{v}_r \cdot \nabla P_r dV_r = P_a \frac{dV_{ol,r}}{dt} + \int_{\partial V_r} \rho_r g w_r dV_r, \tag{B 22}
\]

where \( \mathbf{n}_r \) is a outward pointing unit vector normal to the boundary \( \partial V_r \) enclosing the fluid in the reference state. In Eq. (B 22), the first equality stems from expressing the first integral in the reference state; the second equality uses Eq. (B 19); the third equality results from the integration by part; the final equality stems from that \( P_r \) depends on \( z_r \) and \( t \) only, and that it is in hydrostatic balance, and from using the boundary condition \( \mathbf{v}_r \cdot \mathbf{n}_r = w_r = \partial \eta_r / \partial t \) at the surface assumed to be located at \( z_r = \eta_r(t) \). Having obtained these equations, it is easy to derive equations for \( AGPE \) and \( AIE \) by subtracting

\[
\frac{d(AGPE)}{dt} = W - W_r, \tag{B 23}
\]

\[
\frac{d(AIE)}{dt} = H - B + W_r \tag{B 24}
\]

\[
\frac{d(APE)}{dt} = W - B + H = -C(APE, KE) + H \tag{B 25}
\]

where we define \( AIE \) by

\[
AIE = IE - IE_r + P_r (V_{ol} - V_{ol,r})
\]

to include the contribution of the work of atmospheric pressure against volume changes, where we have used the notations:

\[
W = \int_V \rho g w dV, \quad W_r = \int_V \rho_r g w_r dV \tag{B 26}
\]

\[
B = \int_V P_r \frac{Dv_r}{Dt} dm \tag{B 27}
\]

\[
H = \int_V \left( 1 - \frac{T_r}{T} \right) \hat{Q} dV \tag{B 28}
\]
Interesting, we note that it is also possible to write the evolution equation for $AGPE$ under the form:

$$\frac{d(AGPE)}{dt} = W - B - (W_r - B) = C(KE, APE) - (W_r - B)$$  \hspace{1cm} (B29)

which is now very similar to the expression obtained in the context of the Boussinesq approximation presented in Appendix A.

**B.4. Further partitioning of internal energy into a “dead” and “exergy” component**

An important remark about Lorenz (1955)’s APE is that it is available only for reversible transformations. In stratified fluids, however, a significant fraction of the potential energy is also available for irreversible transformations, as occurs in a laminar fluid when $IE_r$ and $GPE_r$ are exchanged until reaching thermodynamic equilibrium. This motivates defining a notional isothermal state, called the dead state here, possessing exactly the same total energy as the Lorenz (1955)’s reference state, and hence defined by the equation:

$$IE_r + GPE_r + P_a V_{ol,r} = IE_0 + GPE_0 + P_a V_{ol,0}.$$  \hspace{1cm} (B30)

Because both Lorenz’s reference state, as well as any thermodynamic equilibrium state are in hydrostatic balance, the above equality is equivalent to stating that:

$$HE_r = HE_0,$$

where $HE_r$ and $HE_0$ are the total enthalpy of the Lorenz’s reference state and isothermal dead state respectively. Both the reference and dead states are in hydrostatic balance at all times, and hence entirely determined by their temperature and pressure distribution. This makes it possible to define each parcel by their horizontal coordinates $(x, y)$ and hydrostatic pressure $P$, and to assume that the dead state can be obtained from Lorenz’s reference state by an isobaric process, so that $(x_0, y_0, P_0) = (x_r, y_r, P_r)$, which in turns implies $(dx_0/dt, dy_0/dt, dP_0/dt) = (dx_r/dt, dy_r/dt, dP_r/dt)$.

Prior to looking at the evolution of the dead state, let us establish the following result:

$$\int_V \frac{DP}{Dt} dV = \int_V \mathbf{u} \cdot \nabla_h P dV$$  \hspace{1cm} (B31)

when $P$ is in hydrostatic balance, where $\mathbf{u}$ is the horizontal part of the 3D velocity field, and $\nabla_h$ the horizontal part of the nabla operator. The demonstration is:

$$\int_V \left( \frac{DP}{Dt} - \mathbf{u} \cdot \nabla_h P \right) dV = \int_V \left( \frac{\partial P}{\partial t} + \mathbf{u} \cdot \nabla P \right) dV = \frac{d}{dt} \int_V P dV - \frac{d}{dt} \int_V P_a dV_{ol} - \frac{d}{dt} \int_V \rho g z dV = \frac{d}{dt} \left\{ P_a V_{ol} + gM_{tot} + \int_V \rho g z dV \right\} - \frac{d}{dt} \int_V P_a dV_{ol} - \frac{d}{dt} \int_V \rho g z dV = 0$$

where $V_{ol}$ and $M_{tot}$ are the total volume and mass of the fluid, where the expression between brackets was obtained by an integration by parts. The important consequence is that the volume integral of $DP/Dt$ identically vanishes when $P$ is independent of the horizontal coordinates, as is the case for $P_r$ and $P_0$, which we now use to simplify the evolution equation for the total energy of Lorenz (1955)’s reference state:

$$\frac{d(IE_r + GPE_r + P_a V_{ol,r})}{dt} = \frac{d(HE_r)}{dt} = \int_{V_r} \left( T_r \frac{D\Sigma_r}{Dt} + \nu_r \frac{DP_r}{Dt} \right) dV_r$$

$$\int_V \frac{\rho T_r}{T} \dot{Q} dV = \int_V \rho \dot{Q} dV + \int_V \rho \left( \frac{T_r - T}{T} \right) \dot{Q} dV = D + Q_{heating} - Q_{cooling} - H$$  \hspace{1cm} (B32)
by using the fact that the total differential for specific enthalpy $H(\Sigma, P)$ is $dH = T\, d\Sigma + \nu dP$. We can now derive the evolution equation for $HE_0$ as follows:

$$\frac{d(HE_0)}{dt} = \int_{V_0} \rho_0 \left( T_0 \frac{D\Sigma_0}{Dt} + v_0 \frac{DP_0}{Dt} \right) dV_0 = T_0 \int_{V_0} \rho_0 \frac{D\Sigma_0}{Dt} dV_0 \quad (B\,33)$$

which naturally provides the following evolution equation for $\Sigma_0$:

$$\int_{V_0} \rho_0 \frac{D\Sigma_0}{Dt} dV_0 = \frac{D + Q_{heating} - Q_{cooling} - H}{T_0}. \quad (B\,34)$$

This integral yields, using the relation $d\Sigma_0 = C_{\rho_0} dT_0 / T_0 - \alpha_0 / \rho_0 dP_0$,

$$\int_{V_0} \rho_0 C_{\rho_0} \frac{dT_0}{dt} dV_0 - \int_{V_0} \alpha_0 \frac{DP_0}{Dt} dV_0 = \frac{D + Q_{heating} - Q_{cooling} - H}{T_0} \quad (B\,35)$$

Using a mass conservation argument based on writing the mass conservation equation in pressure coordinates as follows:

$$\nabla P_0 \cdot \mathbf{u}_0 + \frac{\partial}{\partial P_0} \left( \frac{DP_0}{Dt} \right) = 0, \quad (B\,36)$$

e.g. Haltiner & Williams (1980), where $\nabla P_0$ is the isobaric horizontal gradient, it is easily shown that the surface integral of $DP_0 / Dt$ vanishes at each $z_0$ or $P_0$ levels. The consequence is that the second term in Eq. (B\,35) vanishes. As a result, Eq. (B\,35) can be rewritten as an equation for $T_0$ as follows:

$$\frac{dT_0}{dt} = \frac{D + Q_{heating} - Q_{cooling} - H}{\int_{V_0} \rho_0 C_{\rho_0} dV_0}. \quad (B\,37)$$

Being in possession of an equation for $T_0$, we now compute the following term by regarding $v_0$ as a function of $T_0$ and $P_0$, using $\rho_0 v_0 = \alpha_0 dT_0 + \rho_0 \partial P_0 v_0 dP_0$,

$$\int_{V_0} \rho_0 P_0 \frac{dV_0}{dt} = \int_{V_0} P_0 \alpha_0 \frac{dV_0}{dt} = \gamma_0 (D + Q_{heating} - Q_{cooling} - H), \quad (B\,38)$$

where we have again used the fact that the surface integral of $DP_0 / Dt$ vanishes at every $z_0$ and $P_0$ levels to simplify the result, and where we defined:

$$\gamma_0 = \frac{\int_{V_0} P_0 \alpha_0 dV_0}{\int_{V_0} \rho_0 C_{\rho_0} dV_0}. \quad (B\,39)$$

Now, using the differential $dIE_0 = T_0 d\Sigma_0 - P_0 dv_0$, and combining Eqs. (B\,34) and (B\,38), one obtains the following equation for the dead part of the IE:

$$\frac{d(IE_0)}{dt} = (1 - \gamma_0) (D + Q_{heating} - Q_{cooling} - H). \quad (B\,40)$$

By subtracting this equation from Eq. (B\,21), we obtain the following evolution equation for the exergy part of the IE:

$$\frac{d(IE_r - IE_0)}{dt} = \gamma_0 (D + Q_{heating} - Q_{cooling} - H) - W_r - P_a \frac{dV_{ol,r}}{dt}. \quad (B\,41)$$

REFERENCES

Energetics of turbulent mixing and the ocean heat engine controversy